Power Flow Analysis

Well known as : Load Flow

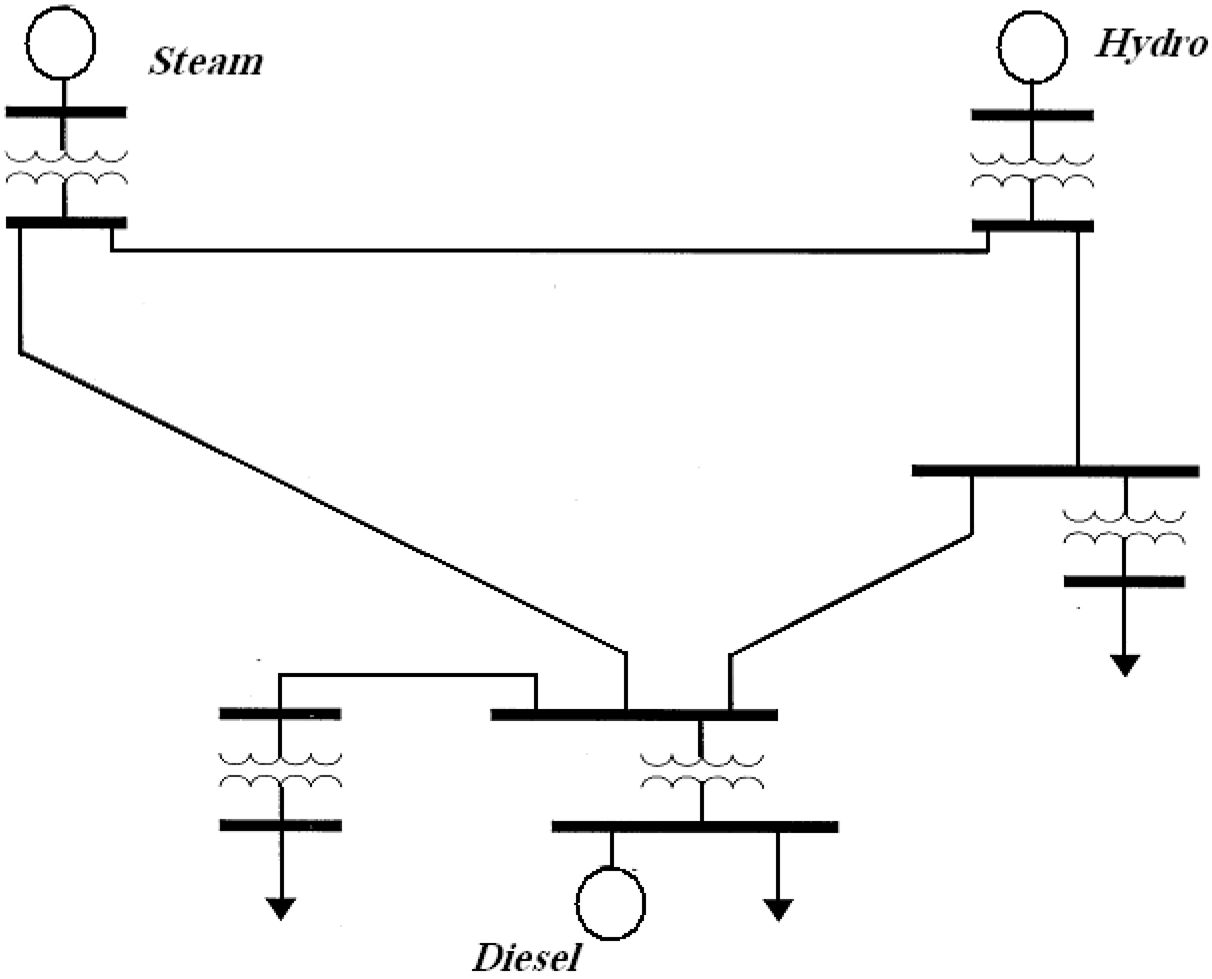
The Power Flow Problem

- Power flow analysis is fundamental to the study of power systems.
- In fact, power flow forms the core of power system analysis.
- power flow study plays a key role in the planning of additions or expansions to transmission and generation facilities.
- A power flow solution is often the starting point for many other types of power system analyses.
- In addition, power flow analysis is at the heart of contingency analysis and the implementation of real-time monitoring systems.

Problem Statement

For a given power network, with known complex power loads and some set of specifications or restrictions on power generations and voltages, **solve** for any unknown bus voltages and unspecified generation and finally for the complex power flow in the network components.

Network Structure



Power Flow Study Steps

1. Determine element values for passive network components.
2. Determine locations and values of all complex power loads.
3. Determine generation specifications and constraints.
4. Develop a mathematical model describing power flow in the network.
5. Solve for the voltage profile of the network.
6. Solve for the power flows and losses in the network.
7. Check for constraint violations.

Formulation of the Bus Admittance Matrix

- The first step in developing the mathematical model describing the power flow in the network is the formulation of the bus admittance matrix.
- The bus admittance matrix is an n*n matrix (where n is the number of buses in the system) constructed from the admittances of the equivalent circuit elements of the segments making up the power system.
- Most system segments are represented by a combination of shunt elements (connected between a bus and the reference node) and series elements (connected between two system buses).

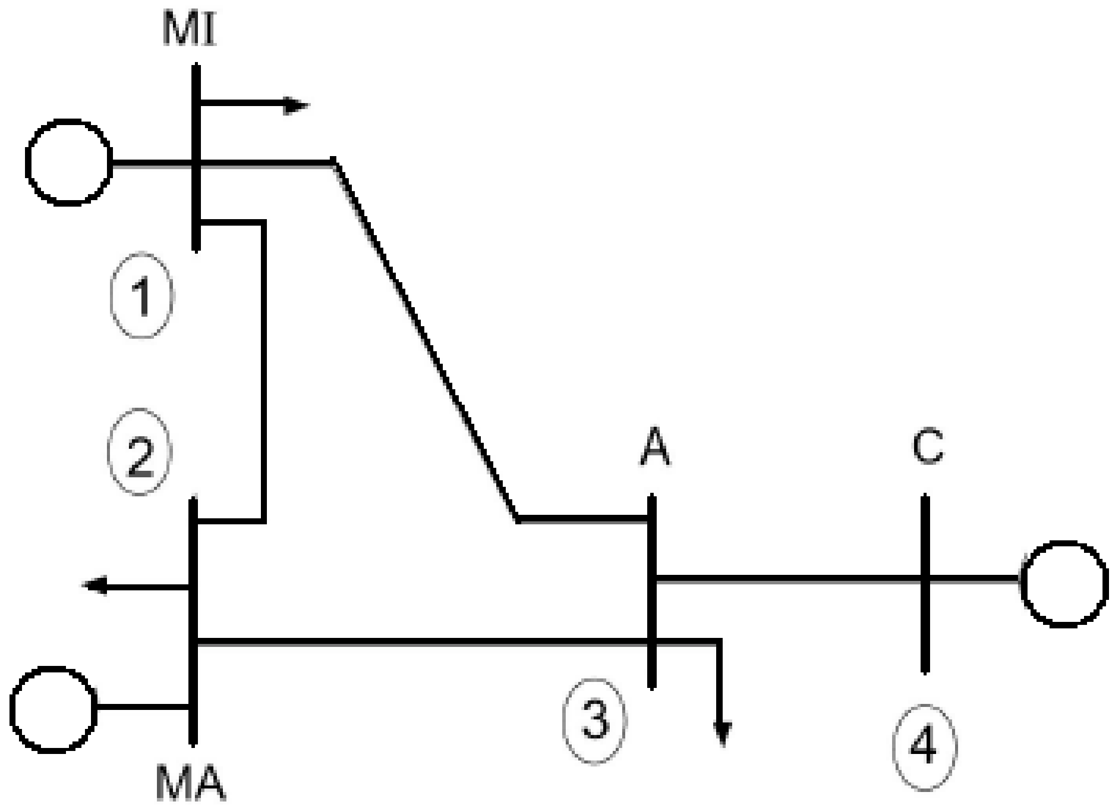
Bus Admittance Matrix

Formulation of the bus admittance matrix follows two simple rules:

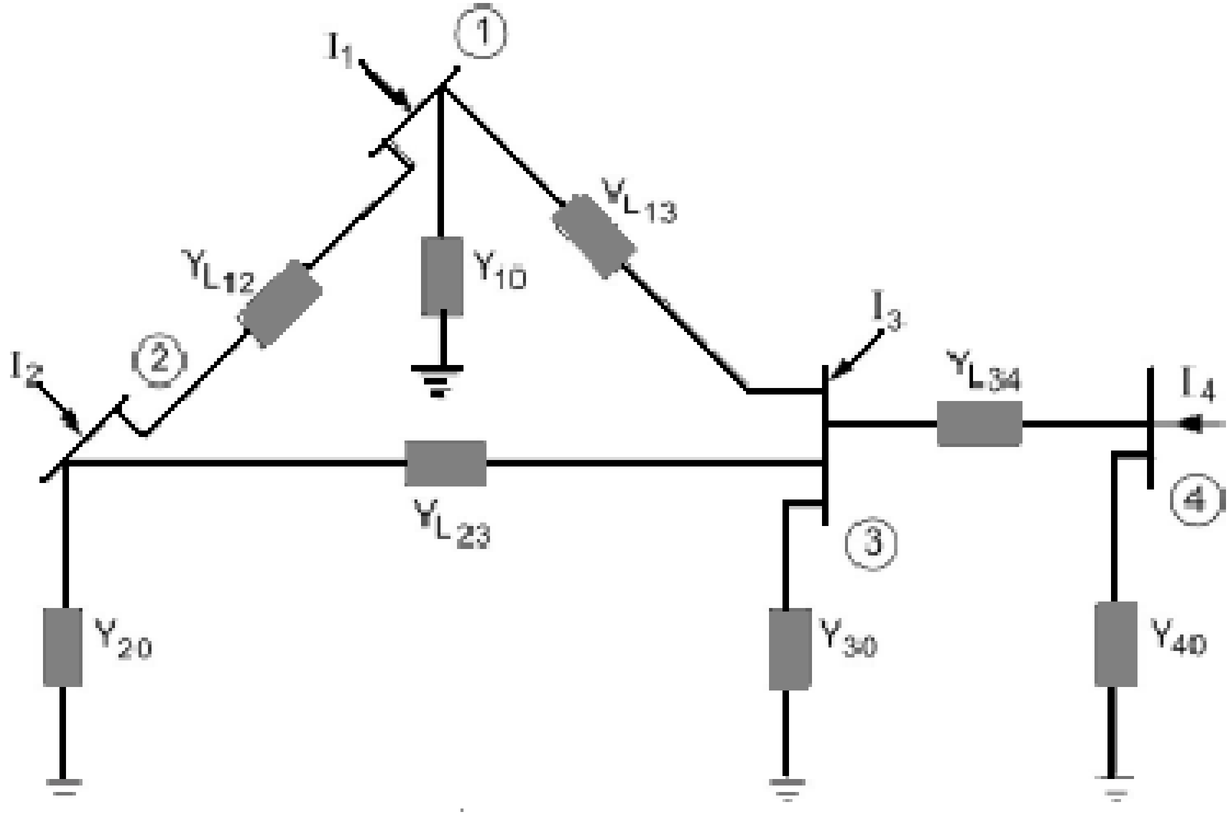
1. The admittance of elements connected between node k and reference is added to the (k, k) entry of the admittance matrix.
2. The admittance of elements connected between nodes j and k is added to the (j, j) and (k, k) entries of the admittance matrix.

- The negative of the admittance is added to the (j, k) and (k, j) entries of the admittance matrix.

Bus Admittance Matrix



Bus Admittance Matrix



Node-Voltage Equations

Applying KCL at each node yields:

$$I_1 = V_1Y_{10} + (V_1 - V_2)Y_{L_{12}} + (V_1 - V_3)Y_{L_{13}}$$
$$I_2 = V_2Y_{20} + (V_2 - V_1)Y_{L_{12}} + (V_2 - V_3)Y_{L_{23}}$$
$$I_3 = V_3Y_{30} + (V_3 - V_1)Y_{L_{13}} + (V_3 - V_4)Y_{L_{34}} + (V_3 - V_2)Y_{L_{23}}$$
$$I_4 = V_4Y_{40} + (V_4 - V_3)Y_{L_{34}}$$

Defining the Y's as

$$Y_{11} = Y_{10} + Y_{L_{12}} + Y_{L_{13}}$$
$$Y_{22} = Y_{20} + Y_{L_{12}} + Y_{L_{23}}$$
$$Y_{33} = Y_{30} + Y_{L_{13}} + Y_{L_{23}} + Y_{L_{34}}$$
$$Y_{44} = Y_{40} + Y_{L_{34}}$$
$$Y_{12} = Y_{21} = -Y_{L_{12}}$$
$$Y_{13} = Y_{31} = -Y_{L_{13}}$$
$$Y_{23} = Y_{32} = -Y_{L_{23}}$$
$$Y_{34} = Y_{43} = -Y_{L_{34}}$$

The Y-Bus

The current equations reduced to

$$I_1 = Y_{11}V_1 + Y_{12}V_2 + Y_{13}V_3 + 0V_4$$
$$I_2 = Y_{21}V_1 + Y_{22}V_2 + Y_{23}V_3 + 0V_4$$
$$I_3 = Y_{13}V_1 + Y_{23}V_2 + Y_{33}V_3 + Y_{34}V_4$$
$$I_4 = 0V_1 + 0V_2 + Y_{43}V_3 + Y_{44}V_4$$

Where,

$$\mathbf{I}_{bus} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} \qquad \mathbf{V}_{bus} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

In a compact form

$$\mathbf{I}_{bus} = \mathbf{Y}_{bus}\mathbf{V}_{bus}$$

$$\mathbf{Y}_{bus} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{12} & Y_{22} & Y_{23} & Y_{24} \\ Y_{13} & Y_{23} & Y_{33} & Y_{34} \\ Y_{14} & Y_{24} & Y_{34} & Y_{44} \end{bmatrix}$$

Gauss Power Flow

We first need to put the equation in the correct form

$$S_i = V_i I_i^* = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k \right)^* = V_i \sum_{k=1}^{n} Y_{ik}^* V_k^*$$

$$S_i^* = V_i^* I_i = V_i^* \sum_{k=1}^{n} Y_{ik} V_k = V_i^* \sum_{k=1}^{n} Y_{ik} V_k$$

$$\frac{S_i^*}{V_i^*} = \sum_{k=1}^{n} Y_{ik} V_k = Y_{ii} V_i + \sum_{k=1, k \neq i}^{n} Y_{ik} V_k$$

$$V_i = \frac{1}{Y_{ii}} \left(\frac{S_i^*}{V_i^*} - \sum_{k=1, k \neq i}^{n} Y_{ik} V_k \right)$$

Difficulties

- Unless the generation equals the load at every bus, the complex power outputs of the generators cannot be arbitrarily selected.
- In fact, the complex power output of at least one of the generators must be calculated last, since it must take up the unknown “slack” due to the uncalculated network losses.
- Further, losses cannot be calculated until the voltages are known.
- Also, it is not possible to solve these equations for the absolute phase angles of the phasor voltages. This simply means that the problem can only be solved to some arbitrary phase angle reference.

Difficulties

- For a 4- bus system, suppose that S_{G4} is arbitrarily allowed to float or swing (in order to take up the necessary slack caused by the losses) and that $S_{G1,}$ $S_{G2,}$ S_{G3} are specified.

$$\overline{S}_{G1}^{*} - \overline{S}_{D1}^{*} = \overline{V}_{1}^{*}\left[\overline{Y}_{11}\overline{V}_{1} + \overline{Y}_{12}\overline{V}_{2} + \overline{Y}_{13}\overline{V}_{3} + \overline{Y}_{14}\overline{V}_{4}\right]$$

$$\overline{S}_{G2}^{*} - \overline{S}_{D2}^{*} = \overline{V}_{2}^{*}\left[\overline{Y}_{21}\overline{V}_{1} + \overline{Y}_{22}\overline{V}_{2} + \overline{Y}_{23}\overline{V}_{3} + \overline{Y}_{24}\overline{V}_{4}\right]$$

$$\overline{S}_{G3}^{*} - \overline{S}_{D3}^{*} = \overline{V}_{3}^{*}\left[\overline{Y}_{31}\overline{V}_{1} + \overline{Y}_{32}\overline{V}_{2} + \overline{Y}_{33}\overline{V}_{3} + \overline{Y}_{34}\overline{V}_{4}\right]$$

$$\overline{S}_{G4}^{*} - \overline{S}_{D4}^{*} = \overline{V}_{4}^{*}\left[\overline{Y}_{41}\overline{V}_{1} + \overline{Y}_{42}\overline{V}_{2} + \overline{Y}_{43}\overline{V}_{3} + \overline{Y}_{44}\overline{V}_{4}\right]$$

Remedies

- Now, with the loads known, the equations are seen as four simultaneous nonlinear equations with complex coefficients in five unknowns. (V_1, V_2, V_3, V_4 and S_{G4}).

- Designating bus 4 as the slack bus and specifying the voltage V_4 reduces the problem to four equations in four unknowns.

Remedies

- The slack bus is chosen as the phase reference for all phasor calculations, its magnitude is constrained, and the complex power generation at this bus is free to take up the slack necessary in order to account for the system real and reactive power losses.

- Systems of nonlinear equations, cannot (except in rare cases) be solved by closed-form techniques.

Load Flow Solution

- There are four quantities of interest associated with each bus:

  1. **Real Power, P**
  2. **Reactive Power, Q**
  3. **Voltage Magnitude, V**
  4. **Voltage Angle, δ**

- At every bus of the system, two of these four quantities will be specified and the remaining two will be unknowns.
- Each of the system buses may be classified in accordance with which of the two quantities are specified

Bus Classifications

Slack Bus — The slack bus for the system is a single bus for which the voltage magnitude and angle are specified.

- The real and reactive power are unknowns.
- The bus selected as the slack bus must have a source of both real and reactive power, since the injected power at this bus must "swing" to take up the "slack" in the solution.
- The best choice for the slack bus (since, in most power systems, many buses have real and reactive power sources) requires experience with the particular system under study.
- The behavior of the solution is often influenced by the bus chosen.

Bus Classifications

Load Bus (P-Q Bus) : A load bus is defined as any bus of the system for which the real and reactive power are specified.

- Load buses may contain generators with specified real and reactive power outputs;
- however, it is often convenient to designate any bus with specified injected complex power as a load bus.

Voltage Controlled Bus (P-V Bus) : Any bus for which the voltage magnitude and the injected real power are specified is classified as a voltage controlled (or P-V) bus.

- The injected reactive power is a variable (with specified upper and lower bounds) in the power flow analysis.
- (A P-V bus must have a variable source of reactive power such as a generator.)

Solution Methods

- The solution of the simultaneous nonlinear power flow equations requires the use of iterative techniques for even the simplest power systems.
- There are many methods for solving nonlinear equations, such as:

- Gauss Seidel.
- Newton Raphson.
- Fast Decoupled.

Guess Solution

- It is important to have a good approximation to the load-flow solution, which is then used as a starting estimate (or initial guess) in the iterative procedure.
- A fairly simple process can be used to evaluate a good approximation to the unknown voltages and phase angles.
- The process is implemented in two stages: the first calculates the approximate angles, and the second calculates the approximate voltage magnitudes.

Gauss Iteration Method

With the Gauss method we need to rewrite our equation in an implicit form: $x = h(x)$

To iterate we first make an initial guess of x, $x^{(0)}$, and then iteratively solve $x^{(v+1)} = h(x^{(v)})$ until we find a "fixed point", $\hat{x}$, such that $\hat{x} = h(\hat{x})$.

Gauss Iteration Example

Example: Solve $x - \sqrt{x} - 1 = 0$

$$x^{(v+1)} = 1 + \sqrt{x^{(v)}}$$

Let k = 0 and arbitrarily guess $x^{(0)} = 1$ and solve

k	$x^{(v)}$	k	$x^{(v)}$
0	1	5	2.61185
1	2	6	2.61612
2	2.41421	7	2.61744
3	2.55538	8	2.61785
4	2.59805	9	2.61798

Stopping Criteria

A key problem to address is when to stop the iteration. With the Guass iteratio we stop when

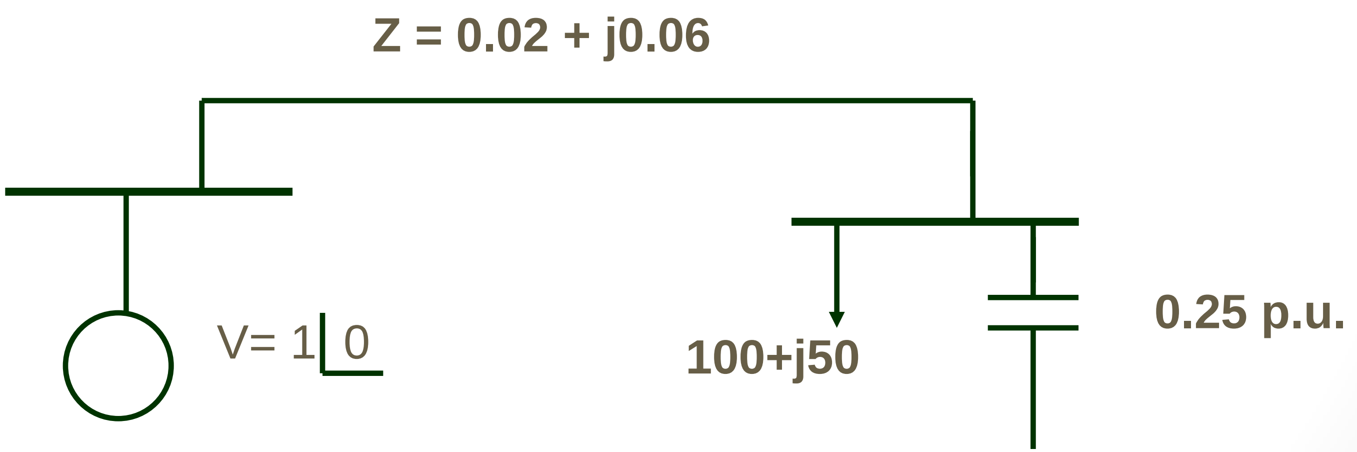
$$\left|\Delta x^{(v)}\right| < \varepsilon \qquad \text{with} \quad \Delta x^{(v)} \triangleq x^{(v+1)} - x^{(v)}$$

If x is a scalar this is clear, but if x is a vector we need to generalize the absolute value by using a norm

$$\left\|\Delta x^{(v)}\right\|_j < \varepsilon$$

Example

A 100 MW, 50 Mvar load is connected to a generator through a line with z = 0.02 + j0.06 p.u. and line charging of 0.05 p.u on each end (100 MVA base). Also, there is a 0.25 p.u. capacitance at bus 2. If the generator voltage is 1.0 p.u., what is V2?



Y-Bus

The unknown is the complex load voltage, V_2.

To determine V_2 we need to know the $\mathbf{Y}_{bus}$.

$$\frac{1}{0.02 + j0.06} = 5 - j15$$

Hence $\mathbf{Y}_{bus} = \begin{bmatrix} 5 - j14.95 & -5 + j15 \\ -5 + j15 & 5 - j14.70 \end{bmatrix}$

(Note $B_{22} = - j15 + j0.05 + j0.25$)

Solution

$$V_2 = \frac{1}{Y_{22}}\left(\frac{S_2^*}{V_2^*} - \sum_{k=1, k\neq i}^{n} Y_{ik}V_k\right)$$

$$V_2 = \frac{1}{5 - j14.70}\left(\frac{-1 + j0.5}{V_2^*} - (-5 + j15)(1.0\angle 0)\right)$$

Guess $V_2^{(0)} = 1.0\angle 0$ (this is known as a flat start)

v	$V_2^{(v)}$	v	$V_2^{(v)}$
0	$1.000 + j0.000$	3	$0.9622 - j0.0556$
1	$0.9671 - j0.0568$	4	$0.9622 - j0.0556$
2	$0.9624 - j0.0553$		

Solution (cont.)

$$V_2 = 0.9622 - j0.0556 = 0.9638\angle -3.3°$$

Once the voltages are known all other values can be determined, such as the generator powers and the line flows

$$S_1^* = V_1^*(Y_{11}V_1 + Y_{12}V_2) = 1.023 - j0.239$$

In actual units $P_1 = 102.3$ MW, $Q_1 = 23.9$ Mvar

Gauss-Seidel Iteration

Immediately use the new voltage estimates:

$$
\begin{aligned}
V_2^{(v+1)} &= h_2(V_1, V_2^{(v)}, V_3^{(v)}, \ldots, V_n^{(v)}) \\
V_3^{(v+1)} &= h_2(V_1, V_2^{(v+1)}, V_3^{(v)}, \ldots, V_n^{(v)}) \\
V_4^{(v+1)} &= h_2(V_1, V_2^{(v+1)}, V_3^{(v+1)}, V_4^{(v)} \ldots, V_n^{(v)}) \\
&\vdots \\
V_n^{(v+1)} &= h_2(V_1, V_2^{(v+1)}, V_3^{(v+1)}, V_4^{(v+1)} \ldots, V_n^{(v)})
\end{aligned}
$$

The Gauss-Seidel works better than the Gauss, and is actually easier to implement. It is used instead of Gauss.

Newton-Raphson Power Flow

The General Form of the Load-Flow Equations

- In Practice, bus powers S_i *is specified* rather than the bus currents I_i .

$$I^* = \frac{S_i}{V_i}$$

- As a result, we have

$$P_i - jQ_i = V_i^* I_i = V^*_i \sum_{n=1}^{N} (Y_{in} V_n) = \sum_{n=1}^{N} |Y_{in} V_i V_n| \angle(\theta_{in} + \delta_n - \delta_i)$$

$$Y_{ij} = |Y_{ij}| \angle \theta_{ij} = |Y_{ij}| \cos\theta_{ij} + j|Y_{ij}| \sin\theta_{ij} = G_{ij} + jB_{ij}$$

Load-Flow Equations

- These are the static power flow equations. Each equation is complex, and therefore we have 2n real equations. The nodal admittance matrix current equation can be written in the power form:

$$P_i - jQ_i = V_i^* I_i = V^*_i \sum_{n=1}^{N} (Y_{in} V_n) = \sum_{n=1}^{N} |Y_{in} V_i V_n| \angle(\theta_{in} + \delta_n - \delta_i)$$

Let,

$$Y_{ij} = |Y_{ij}| \angle \theta_{ij} = |Y_{ij}| \cos\theta_{ij} + j|Y_{ij}| \sin\theta_{ij} = G_{ij} + jB_{ij}$$

Load-Flow Equations

- **Finally,**

$$P_i = |V_i|^2 G_{ii} + \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \cos(\theta_{in} + \delta_n - \delta_i)$$

$$Q_i = -|V_i|^2 B_{ii} - \sum_{\substack{n=1 \\ n \neq i}}^{N} |V_i V_n Y_{in}| \sin(\theta_{in} + \delta_n - \delta_i)$$

- **This is known as NR (Newton – Raphson) formulation**

Newton-Raphson Algorithm

- The second major power flow solution method is the Newton-Raphson algorithm.
- Key idea behind Newton-Raphson is to use sequential linearization

General form of problem: Find an x such that

$$f(\hat{x}) = 0$$

Newton-Raphson Method

1. For each guess of $\hat{x}$, $x^{(v)}$, define

   $\Delta x^{(v)} = \hat{x} - x^{(v)}$

2. Represent $f(\hat{x})$ by a Taylor series about $f(x)$

$$f(\hat{x}) = f(x^{(v)}) + \frac{df(x^{(v)})}{dx}\Delta x^{(v)} + $$

$$+ \frac{1}{2}\frac{d^2 f(x^{(v)})}{dx^2}\left(\Delta x^{(v)}\right)^2 + \text{higher order terms}$$

Newton-Raphson Method

3. Approximate $f(\hat{x})$ by neglecting all terms except the first two

$$f(\hat{x}) = 0 \approx f(x^{(v)}) + \frac{df(x^{(v)})}{dx} \Delta x^{(v)}$$

4. Use this linear approximation to solve for $\Delta x^{(v)}$

$$\Delta x^{(v)} = -\left[\frac{df(x^{(v)})}{dx}\right]^{-1} f(x^{(v)})$$

5. Solve for a new estimate of x̂

$$x^{(v+1)} = x^{(v)} + \Delta x^{(v)}$$

Example

Use Newton-Raphson to solve $f(x) = x^2 - 2 = 0$

The equation we must iteratively solve is

$$\Delta x^{(v)} = -\left[\frac{df(x^{(v)})}{dx}\right]^{-1} f(x^{(v)})$$

$$\Delta x^{(v)} = -\left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

$$x^{(v+1)} = x^{(v)} + \Delta x^{(v)}$$

$$x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

Example Solution

$$x^{(v+1)} = x^{(v)} - \left[\frac{1}{2x^{(v)}}\right]((x^{(v)})^2 - 2)$$

Guess $x^{(0)} = 1$. Iteratively solving we get

v	$x^{(v)}$	$f(x^{(v)})$	$\Delta x^{(v)}$
0	1	-1	0.5
1	1.5	0.25	-0.08333
2	1.41667	6.953×10^{-3}	-2.454×10^{-3}
3	1.41422	6.024×10^{-6}	

Comments

- When close to the solution the error decreases quite quickly -- method has quadratic convergence
- Stopping criteria is when $|f(x^{(v)})| < \varepsilon$
- Results are dependent upon the initial guess. What if we had guessed $x^{(0)} = 0$, or $x^{(0)} = -1$?

Multi-Variable Newton-Raphson

Next we generalize to the case where $\mathbf{x}$ is an n-dimension vector, and $\mathbf{f}(\mathbf{x})$ is an n-dimension function

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix}$$

Again define the solution $\hat{\mathbf{x}}$ so $\mathbf{f}(\hat{\mathbf{x}}) = 0$ and

$$\Delta\mathbf{x} = \hat{\mathbf{x}} - \mathbf{x}$$

Multi-Variable Case, cont'd

The Taylor series expansion is written for each $f_i(\mathbf{x})$

$$
\begin{aligned}
f_1(\hat{\mathbf{x}}) \;&=\; f_1(\mathbf{x}) + \frac{\partial f_1(\mathbf{x})}{\partial x_1}\Delta x_1 + \frac{\partial f_1(\mathbf{x})}{\partial x_2}\Delta x_2 + \ldots \\
&\quad \frac{\partial f_1(\mathbf{x})}{\partial x_n}\Delta x_n + \text{higher order terms}
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
f_n(\hat{\mathbf{x}}) \;&=\; f_n(\mathbf{x}) + \frac{\partial f_n(\mathbf{x})}{\partial x_1}\Delta x_1 + \frac{\partial f_n(\mathbf{x})}{\partial x_2}\Delta x_2 + \ldots \\
&\quad \frac{\partial f_n(\mathbf{x})}{\partial x_n}\Delta x_n + \text{higher order terms}
\end{aligned}
$$

Multi-Variable Case, cont'd

This can be written more compactly in matrix form

$$\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$+ \quad \text{higher order terms}$$

Jacobian Matrix

The n by n matrix of partial derivatives is known as the Jacobian matrix, $\mathbf{J}(\mathbf{x})$

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

Multi-Variable N-R Procedure

Derivation of N-R method is similar to the scalar case

$$\begin{aligned}
\mathbf{f}(\hat{\mathbf{x}}) &= \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\Delta\mathbf{x} + \text{higher order terms} \\
\mathbf{f}(\hat{\mathbf{x}}) &= 0 \approx \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\Delta\mathbf{x} \\
\Delta\mathbf{x} &\approx -\mathbf{J}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}) \\
\mathbf{x}^{(v+1)} &= \mathbf{x}^{(v)} + \Delta\mathbf{x}^{(v)} \\
\mathbf{x}^{(v+1)} &= \mathbf{x}^{(v)} - \mathbf{J}(\mathbf{x}^{(v)})^{-1}\mathbf{f}(\mathbf{x}^{(v)})
\end{aligned}$$

Iterate until $\left\| \mathbf{f}(\mathbf{x}^{(v)}) \right\| < \varepsilon$

Example

Solve for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $\mathbf{f}(\mathbf{x}) = 0$ where

$$\mathrm{f}_1(\mathbf{x}) = 2x_1^2 + x_2^2 - 8 = 0$$

$$\mathrm{f}_2(\mathbf{x}) = x_1^2 - x_2^2 + x_1 x_2 - 4 = 0$$

First symbolically determine the Jacobian

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_1} & \dfrac{\partial \mathrm{f}_1(\mathbf{x})}{\partial x_2} \\ \dfrac{\partial \mathrm{f}_2(\mathbf{x})}{\partial x_1} & \dfrac{\partial \mathrm{f}_2(\mathbf{x})}{\partial x_2} \end{bmatrix}$$

Solution

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} 4x_1 & 2x_2 \\ 2x_1 + x_2 & x_1 - 2x_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = -\begin{bmatrix} 4x_1 & 2x_2 \\ 2x_1 + x_2 & x_1 - 2x_2 \end{bmatrix}^{-1} \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$$

Arbitrarily guess $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 1.3 \end{bmatrix}$$

Solution, cont'd

$$\mathbf{x}^{(2)} = \begin{bmatrix} 2.1 \\ 1.3 \end{bmatrix} - \begin{bmatrix} 8.40 & 2.60 \\ 5.50 & -0.50 \end{bmatrix}^{-1} \begin{bmatrix} 2.51 \\ 1.45 \end{bmatrix} = \begin{bmatrix} 1.8284 \\ 1.2122 \end{bmatrix}$$

Each iteration we check $\|\mathbf{f}(\mathbf{x})\|$ to see if it is below our specified tolerance ε

$$\mathbf{f}(\mathbf{x}^{(2)}) = \begin{bmatrix} 0.1556 \\ 0.0900 \end{bmatrix}$$

If $\varepsilon = 0.2$ then we would be done. Otherwise we'd continue iterating.

NR Application to Power Flow

We first need to rewrite complex power equations as equations with real coefficients

$$S_i = V_i I_i^* = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k \right)^* = V_i \sum_{k=1}^{n} Y_{ik}^* V_k^*$$

These can be derived by defining

$$Y_{ik} = G_{ik} + jB_{ik}$$

$$V_i = |V_i| e^{j\theta_i} = |V_i| \angle \theta_i$$

$$\theta_{ik} = \theta_i - \theta_k$$

Recall $e^{j\theta} = \cos\theta + j\sin\theta$

Power Balance Equations

$$S_i = P_i + jQ_i = V_i \sum_{k=1}^{n} Y_{ik}^* V_k^* = \sum_{k=1}^{n} |V_i||V_k| e^{j\theta_{ik}} (G_{ik} - jB_{ik})$$

$$= \sum_{k=1}^{n} |V_i||V_k| (\cos\theta_{ik} + j\sin\theta_{ik})(G_{ik} - jB_{ik})$$

Resolving into the real and imaginary parts

$$P_i = \sum_{k=1}^{n} |V_i||V_k| (G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$Q_i = \sum_{k=1}^{n} |V_i||V_k| (G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

NR Power Flow

In the Newton-Raphson power flow we use Newton's method to determine the voltage magnitude and angle at each bus in the power system.

We need to solve the power balance equations

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Power Flow Variables

Assume the slack bus is the first bus (with a fixed voltage angle/magnitude). We then need to determine the voltage angle/magnitude at the other buses.

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_\mathrm{n} \\ |V_2| \\ \vdots \\ |V_n| \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_{G2} + P_{D2} \\ \vdots \\ P_n(\mathbf{x}) - P_{Gn} + P_{Dn} \\ Q_2(\mathbf{x}) - Q_{G2} + Q_{D2} \\ \vdots \\ Q_n(\mathbf{x}) - Q_{Gn} + Q_{Dn} \end{bmatrix}$$

N-R Power Flow Solution

The power flow is solved using the same procedure discussed last time:

Set $v = 0$; make an initial guess of $\mathbf{x}$, $\mathbf{x}^{(v)}$

While $\left\| \mathbf{f}(\mathbf{x}^{(v)}) \right\| > \varepsilon$ Do

$$\mathbf{x}^{(v+1)} = \mathbf{x}^{(v)} - \mathbf{J}(\mathbf{x}^{(v)})^{-1}\mathbf{f}(\mathbf{x}^{(v)})$$

$$v = v+1$$

End While

Power Flow Jacobian Matrix

The most difficult part of the algorithm is determining and inverting the n by n Jacobian matrix, $\mathbf{J}(\mathbf{x})$

$$
\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \frac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}
$$

Power Flow Jacobian Matrix,

Jacobian elements are calculated by differentiating each function, $f_i(\mathbf{x})$, with respect to each variable. For example, if $f_i(\mathbf{x})$ is the bus i real power equation

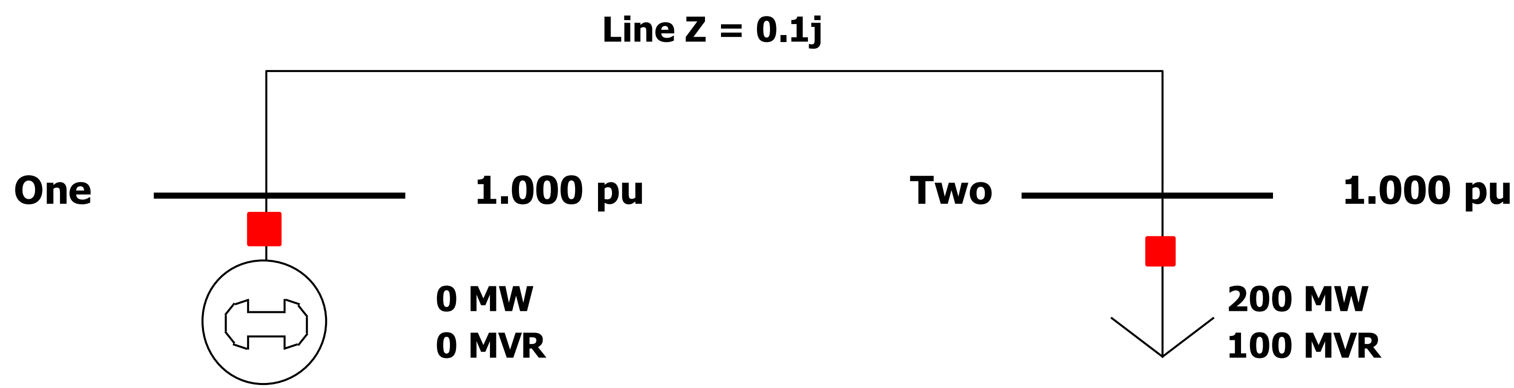
$$f_i(x) = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) - P_{Gi} + P_{Di}$$

$$\frac{\partial f_i(x)}{\partial \theta_i} = \sum_{\substack{k=1 \\ k\neq i}}^{n} |V_i||V_k|(-G_{ik}\sin\theta_{ik} + B_{ik}\cos\theta_{ik})$$

$$\frac{\partial f_i(x)}{\partial \theta_j} = |V_i||V_j|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) \quad (j \neq i)$$

Two Bus, Example

For the two bus power system shown below, use the Newton-Raphson power flow to determine the voltage magnitude and angl at bus two. Assume that bus one is the slack and $S_{Base} = 100$ MVA



$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ |V_2| \end{bmatrix} \qquad \mathbf{Y}_{bus} = \begin{bmatrix} -j10 & j10 \\ j10 & -j10 \end{bmatrix}$$

Two Bus Example, cont'd

General power balance equations

$$\mathrm{P_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\cos\theta_{ik} + B_{ik}\sin\theta_{ik}) = P_{Gi} - P_{Di}$$

$$\mathrm{Q_i} = \sum_{k=1}^{n} |V_i||V_k|(G_{ik}\sin\theta_{ik} - B_{ik}\cos\theta_{ik}) = Q_{Gi} - Q_{Di}$$

Bus two power balance equations

$$\mathrm{P_2} = |V_2||V_1|(10\sin\theta_2) + 2.0 = 0$$

$$Q_2 = |V_2||V_1|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 = 0$$

Two Bus Example, cont'd

$$\mathrm{P}_2(\mathbf{x}) = |V_2|(10\sin\theta_2) + 2.0 = 0$$

$$Q_2(\mathbf{x}) = |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 = 0$$

Now calculate the power flow Jacobian

$$J(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial \mathrm{P}_2(\mathbf{x})}{\partial \theta_2} & \dfrac{\partial \mathrm{P}_2(\mathbf{x})}{\partial |V|_2} \\ \dfrac{\partial \mathrm{Q}_2(\mathbf{x})}{\partial \theta_2} & \dfrac{\partial \mathrm{Q}_2(\mathbf{x})}{\partial |V|_2} \end{bmatrix}$$

$$= \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix}$$

First Iteration

Set $v = 0$, guess $\mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Calculate

$$\mathrm{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} |V_2|(10\sin\theta_2) + 2.0 \\ |V_2|(-10\cos\theta_2) + |V_2|^2(10) + 1.0 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(0)}) = \begin{bmatrix} 10|V_2|\cos\theta_2 & 10\sin\theta_2 \\ 10|V_2|\sin\theta_2 & -10\cos\theta_2 + 20|V_2| \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Solve $\mathbf{x}^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix}$

Next Iterations

$$f(\mathbf{x}^{(1)}) = \begin{bmatrix} 0.9(10\sin(-0.2)) + 2.0 \\ 0.9(-10\cos(-0.2)) + 0.9^2 \times 10 + 1.0 \end{bmatrix} = \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix}$$

$$\mathbf{J}(\mathbf{x}^{(1)}) = \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}$$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -0.2 \\ 0.9 \end{bmatrix} - \begin{bmatrix} 8.82 & -1.986 \\ -1.788 & 8.199 \end{bmatrix}^{-1} \begin{bmatrix} 0.212 \\ 0.279 \end{bmatrix} = \begin{bmatrix} -0.233 \\ 0.8586 \end{bmatrix}$$

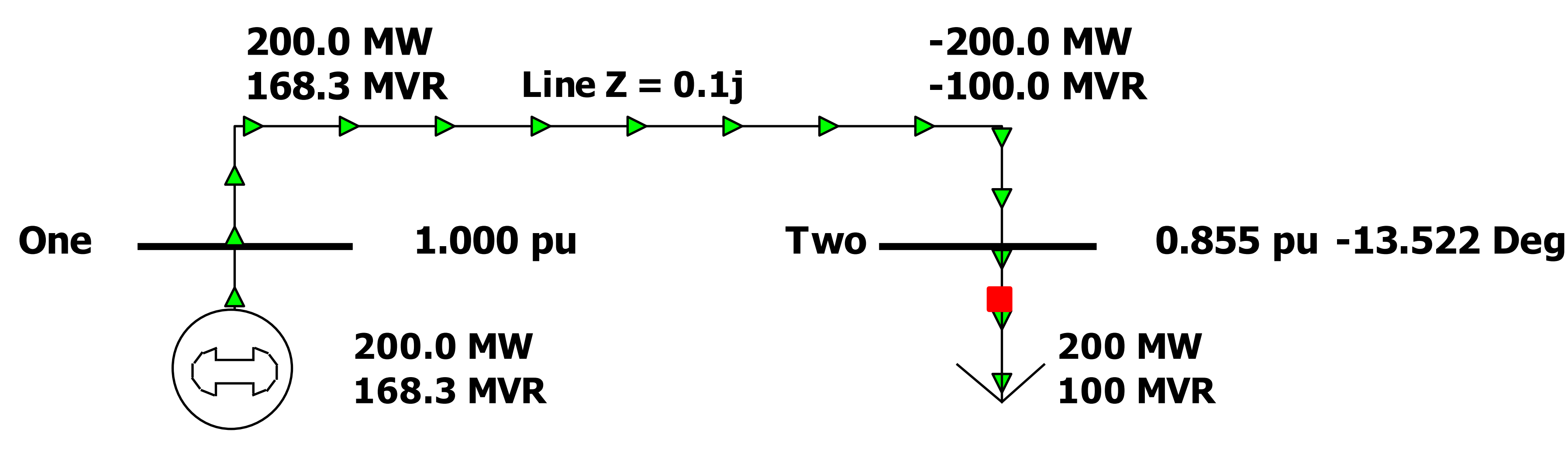
$$f(\mathbf{x}^{(2)}) = \begin{bmatrix} 0.0145 \\ 0.0190 \end{bmatrix} \qquad \mathbf{x}^{(3)} = \begin{bmatrix} -0.236 \\ 0.8554 \end{bmatrix}$$

$$f(\mathbf{x}^{(3)}) = \begin{bmatrix} 0.0000906 \\ 0.0001175 \end{bmatrix}$$

Done! $V_2 = 0.8554\angle -13.52^\circ$

Two Bus Solved Values

Once the voltage angle and magnitude at bus 2 are known we can calculate all the other system values, such as the line flows and the generator reactive power output



PV Buses

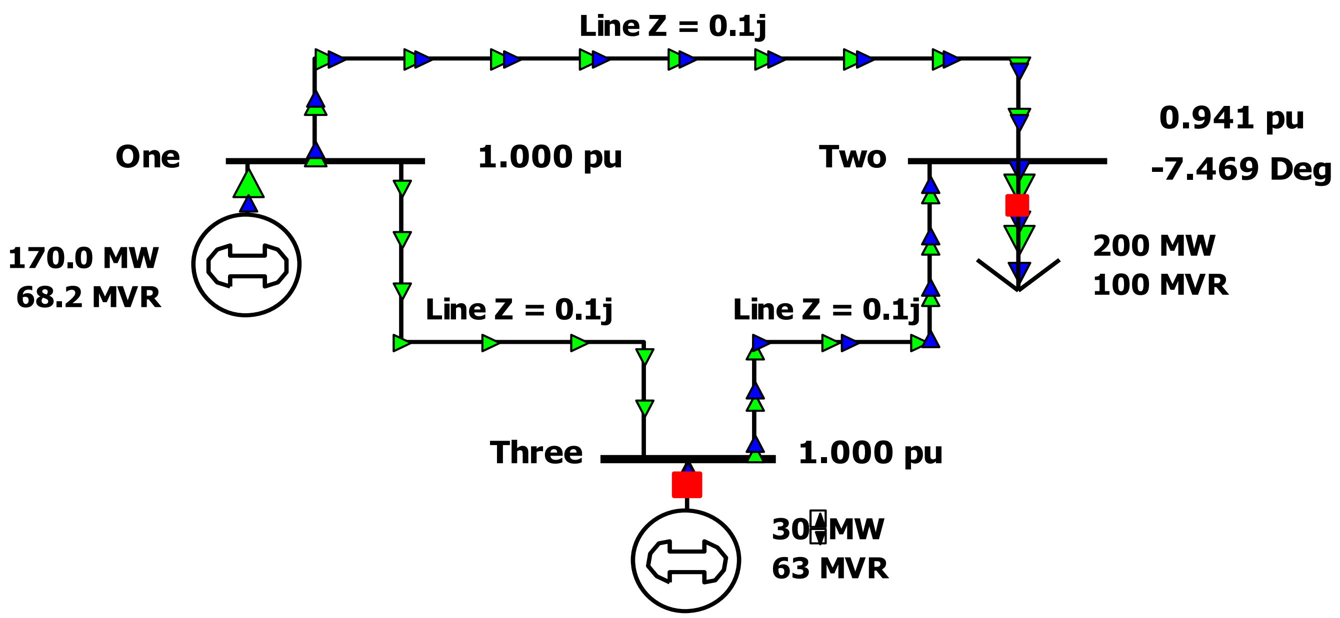
- Since the voltage magnitude at PV buses is fixed there is no need to explicitly include these voltages in **x** or write the reactive power balance equations
  - the reactive power output of the generator varies to maintain the fixed terminal voltage (within limits)
  - optionally these variations/equations can be included by just writing the explicit voltage constraint for the generator bus

$$|V_i| - V_{i\ setpoint} = 0$$

Three Bus PV Case Example

For this three bus case we have

$$\mathbf{x} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ |V_2| \end{bmatrix} \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} P_2(\mathbf{x}) - P_{G2} + P_{D2} \\ P_3(\mathbf{x}) - P_{G3} + P_{D3} \\ Q_2(\mathbf{x}) + Q_{D2} \end{bmatrix} = 0$$



N-R Power Flow

- Advantages
  - fast convergence as long as initial guess is close to solution
  - large region of convergence
- Disadvantages
  - each iteration takes much longer than a Gauss-Seidel iteration
  - more complicated to code, particularly when implementing sparse matrix algorithms
- Newton-Raphson algorithm is very common in power flow analysis