Lecture # 10

- Filter Design By The Insertion Loss Method
  - Design Steps
    - Low-pass prototype design. (✓)
    - Scaling and conversion. (✓)
    - Implementation.
      - Using Stubs.
      - Using High-Low Impedance Sections.
    - Theory of periodic structures.
    - Image impedances and Transfer functions for two-port networks.

Stepped Impedance Low-Pass Filters

By using alternating sections of very high and very low characteristic impedance lines. Known as Stepped-impedance or hi-Z, low-Z filters.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
\cos \beta l & jZ_o \sin \beta l \\
j \sin \beta l & \cos \beta l
\end{bmatrix}
\]

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{22} & Z_{21}
\end{bmatrix} = \begin{bmatrix}
A & AD - BC \\
C & D
\end{bmatrix}
\]

Reciprocal Network  \(AD - BC = 1\)

\[
Z_{11} = Z_{22} = \frac{A}{C} = -jZ_o \cot \beta l
\]

\[
Z_{12} = Z_{21} = \frac{1}{C} = -j \frac{Z_o}{\sin \beta l}
\]

\[
[Z]_\text{inv} = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix} = \begin{bmatrix}
-jZ_o \cot \beta l & -jZ_o \\
-j \frac{Z_o}{\sin \beta l} & -j \frac{Z_o}{\sin \beta l}
\end{bmatrix}
\]
**Stepped Impedance Low-Pass Filters**

\[
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\]

\[Z_{11} - Z_{12} = -jZ_o \left[ \frac{\cos \beta l - 1}{\sin \beta l} \right] = jZ_o \sin(\beta l/2) \cos(\beta l/2) = jZ_o \tan\left(\frac{\beta l}{2}\right) = j \frac{X}{2}\]

\[Z_{12} = Z_{21} = \frac{1}{C} = -j \frac{Z_o}{\sin \beta l} = -j \frac{1}{B}\]

**T-Equivalent Circuit**

\[jB\]

---

**Stepped Impedance Low-Pass Filters**

\[\frac{X}{2} = Z_o \tan\left(\frac{\beta l}{2}\right)\]

\[B = \frac{1}{Z_o} \sin \beta l\]

If we have a short length of line and large characteristic impedance:

\[X \approx Z_o \beta l\]

\[B \approx 0\]

\[Z_o = Z_{high-impedance} = Z_h\]

If we have a short length of line and small characteristic impedance:

\[X \approx 0\]

\[B \approx Y_o \beta l\]

\[Z_o = Z_{low-impedance} = Z_l\]

\[\beta l = \frac{LR_o}{Z_h} \text{ (inductor)}\]

\[\beta l = \frac{CZ_l}{R_o} \text{ (capacitor)}\]

**Ratio should be as high as possible**
Example 8.6

Design a stepped-impedance low-pass filter having a maximally flat response and a cutoff frequency of 2.5 GHz. It is necessary to have more than 20 dB insertion loss at 4 GHz. The filter impedance is 50 Ω; the highest practical line impedance is 120 Ω, and the lowest is 20 Ω. Consider the effect of losses when this filter is implemented with a microstrip substrate having d=0.158 cm, $\varepsilon_r=4.2$, $\tan \delta = 0.02$, and copper conductors of 0.5 mil thickness.

answer

$$\omega_c = 2.5 \text{ GHz} \quad IL = 20 \text{ dB} \quad @ \quad \omega = 4 \text{ GHz} \quad \frac{\omega}{\omega_c} - 1 = \frac{4}{2.5} - 1 = 0.6$$

$$Z_v = R_v = 50 \Omega \quad Z_h = 120 \Omega \quad Z_l = 20 \Omega \quad \frac{Z_h}{Z_l} = 6$$
Example 8.6

\[ g_1 = 0.517 = C_1 \]
\[ g_2 = 1.414 = L_2 \]
\[ g_3 = 1.932 = C_3 \]
\[ g_4 = 1.932 = L_4 \]
\[ g_5 = 1.414 = C_5 \]
\[ g_6 = 0.517 = L_6 \]

\[ \beta l_1 = \frac{C_1 Z_1}{R_0} = \frac{0.517 \times 20}{50} = 0.2068 \rightarrow 1.849^\circ \]
\[ \beta l_2 = \frac{L_2 R_0}{Z_h} = \frac{1.414 \times 50}{120} = 0.5892 \rightarrow 33.759^\circ \]

<table>
<thead>
<tr>
<th>Section</th>
<th>( Z_i = Z_d ) or ( Z_h )</th>
<th>( \beta \ell_i )</th>
<th>( W_i ) (mm)</th>
<th>( \ell_i ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20 Ω</td>
<td>11.8°</td>
<td>11.3</td>
<td>2.05</td>
</tr>
<tr>
<td>2</td>
<td>120 Ω</td>
<td>33.8°</td>
<td>0.428</td>
<td>6.63</td>
</tr>
<tr>
<td>3</td>
<td>20 Ω</td>
<td>44.3°</td>
<td>11.3</td>
<td>7.69</td>
</tr>
<tr>
<td>4</td>
<td>120 Ω</td>
<td>46.1°</td>
<td>0.428</td>
<td>9.04</td>
</tr>
<tr>
<td>5</td>
<td>20 Ω</td>
<td>32.4°</td>
<td>11.3</td>
<td>5.63</td>
</tr>
<tr>
<td>6</td>
<td>120 Ω</td>
<td>12.3°</td>
<td>0.428</td>
<td>2.41</td>
</tr>
</tbody>
</table>
**Theory of periodic structures**

- An infinite transmission line periodically loaded with reactive elements is referred to as a **periodic structure**.

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**Example 8.6**

![Graph showing attenuation vs frequency](image)
Analysis of infinite periodic structures

Let \( \theta = kd \)

Where \( k \) is the propagation constant of the unloaded line

Also assume normalized value to \( Z_o \)

\[
\left[\begin{array}{cc}
A & B \\
C & D
\end{array}\right] = \left[\begin{array}{cc}
\cos kl & jZ_o \sin kl \\
jY_o \sin kl & \cos kl
\end{array}\right] \quad \left[\begin{array}{cc}
A & B \\
C & D
\end{array}\right] = \left[\begin{array}{cc}
1 & 0 \\
Y & 1
\end{array}\right]
\]
Analysis of infinite periodic structures

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
= 
\begin{pmatrix}
\cos \theta - \frac{b}{2} \sin \theta & j \left( \sin \theta + \frac{b}{2} \cos \theta - \frac{b}{2} \right) \\
\sin \theta + \frac{b}{2} \cos \theta + \frac{b}{2} & \cos \theta - \frac{b}{2} \sin \theta
\end{pmatrix}
\]

For a wave propagating in the +ve direction.

\[
V(z) = V(0) e^{-ikz} \quad \text{or} \quad V_{n+1} = V_n e^{-ikd}
\]

\[
I(z) = I(0) e^{-ikz} \quad \text{or} \quad I_{n+1} = I_n e^{-ikd}
\]

\[
\begin{pmatrix}
V_n \\
I_n
\end{pmatrix}
= 
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
V_{n+1} \\
I_{n+1}
\end{pmatrix}
\]

\[
A V_{n+1} + B I_{n+1} = V_n e^{ikd}
\]

\[
C V_{n+1} + D I_{n+1} = I_n e^{ikd}
\]

\[
(A - e^{ikd}) V_{n+1} + B I_{n+1} = 0
\]

\[
C V_{n+1} + (D - e^{ikd}) I_{n+1} = 0
\]

\[
\begin{pmatrix}
A - e^{ikd} & B \\
C & D - e^{ikd}
\end{pmatrix}
\begin{pmatrix}
V_{n+1} \\
I_{n+1}
\end{pmatrix}
= 0
\]
Analysis of infinite periodic structures

For a Reciprocal Network in terms of S-parameters
\[ S_{12} = S_{21} \]

In terms of Transmission parameters, using conversion tables

\[
\frac{2(AD - BC)}{A + B/Z_o + CZ_o + D} = \frac{2}{A + B/Z_o + CZ_o + D} \rightarrow AD - BC = 1
\]

For a non-trivial solution, the determinant of the matrix must vanish

\[
\begin{bmatrix} A - e^{\gamma d} & B \\ C & D - e^{\gamma d} \end{bmatrix} \begin{bmatrix} V_{n+1} \\ I_{n+1} \end{bmatrix} = 0 \rightarrow (A - e^{\gamma d})(D - e^{\gamma d}) - BC = 0
\]

\[
AD + e^{2\gamma d} - (A + D)e^{\gamma d} - BC = 0 \rightarrow 1 + e^{2\gamma d} - (A + D)e^{\gamma d} = 0
\]

Divide by \( e^{\gamma d} \)

\[
e^{-\gamma d} + e^{\gamma d} - (A + D) = 0 \rightarrow e^{-\gamma d} + e^{\gamma d} = A + D
\]

But \( \cosh \gamma d = \frac{e^{-\gamma d} + e^{\gamma d}}{2} \) and \( A = D = \cos \theta - \frac{b}{2} \sin \theta \) hence

Analysis of infinite periodic structures

\[
\cosh \gamma d = \frac{A + D}{2} = \cos \theta - \frac{b}{2} \sin \theta
\]

Assume \( \gamma = \alpha + j\beta \)

Thus \( e^{\gamma d} = \cosh \alpha d \cos \beta d + j \sinh \alpha d \sin \beta d = \frac{A + D}{2} = \cos \theta - \frac{b}{2} \sin \theta \)

Accordingly, must have either \( \alpha = 0 \), or \( \beta = 0 \)

**Case 1** \( \alpha = 0 \), \( \beta \neq 0 \)

Non-attenuating (pass-band) (Complex \( \gamma \))

\( \cos \beta d = \cos \theta - \frac{b}{2} \sin \theta \)

Since \( |\cos \beta d| \leq 1 \)

\( |\cos \theta - \frac{b}{2} \sin \theta| \leq 1 \)

**Case 2** \( \alpha \neq 0 \), \( \beta = 0, \pi \)

Attenuating (stop-band) (Real \( \gamma \))

\( \cosh \alpha d = \cos \theta - \frac{b}{2} \sin \theta \)

Since \( |\cosh \alpha d| \geq 1 \)

\( |\cos \theta - \frac{b}{2} \sin \theta| \geq 1 \)

**Note** line is lossless, so power is not dissipated it is just reflected back to the input.
Analysis of infinite periodic structures

Attenuating (stop-band) 
(Real $\gamma$)

Non-attenuating (pass-band) 
(Complex $\gamma$)
Analysis of infinite periodic structures

Characteristic impedance (Bloch Impedance) at the unit cell terminals is given by

\[ Z_B = Z_0 \frac{V_{n+1}}{I_{n+1}} \]

Note: the values were normalized

From the determinant of the matrix

\[ e^{2\beta d} - (A + D)e^{\beta d} + 1 = 0 \rightarrow x^2 - (A + D)x + 1 = 0 \rightarrow e^{\beta d} = \frac{(A + D) \pm \sqrt{(A + D)^2 - 4}}{2} \]

\[ Z_B^* = \frac{-2BZ_0}{2A - A - D + \sqrt{(A + D)^2 - 4}} \rightarrow A = D \]

For symmetrical cells

\[ Z_B^* = \frac{\pm BZ_0}{\sqrt{A^2 - 1}} \]

\[ + \text{ For positive traveling waves.} \]

\[ - \text{ For negative traveling waves.} \]
Analysis of infinite periodic structures

Note that B is always imaginary

\[ Z_B^\pm = \pm \frac{BZ_o}{\sqrt{A^2 - 1}} \]

Hence

\[ B = j \left( \sin \theta + \frac{b}{2} \cos \theta - \frac{b}{2} \right) \]
\[ A = \cos \theta - \frac{b}{2} \sin \theta \]

Case 1: \( \alpha = 0, \beta \neq 0 \)
Non-attenuating (pass-band)

\[ \left| \cos \theta - \frac{b}{2} \sin \theta \right| \leq 1 \]
\[ \sqrt{A^2 - 1} \quad \text{Imaginary} \]

\[ Z_B^\pm = \text{Real} \]

Similar to Propagating Modes

Case 2: \( \alpha \neq 0, \beta = 0, \pi \)
Attenuating (stop-band)

\[ \left| \cos \theta - \frac{b}{2} \sin \theta \right| \geq 1 \]
\[ \sqrt{A^2 - 1} \quad \text{Real} \]

\[ Z_B^\pm = \text{Imaginary} \]

Similar to Evanescent modes (non propagating modes)

Terminated Periodic Structures

The incident and reflected voltages at the \( n^{th} \) unit cell

\[ V_n^+ = V_o e^{-j\beta nd} + V_o e^{j\beta nd} \]

\[ V_n^- = V_o e^{-j\beta nd} + V_o e^{j\beta nd} \]

\[ I_n = I_n^+ e^{-j\beta nd} + I_n^- e^{j\beta nd} = \frac{V_o}{Z_B} e^{-j\beta nd} + \frac{V_o}{Z_B} e^{j\beta nd} \]

\[ I_n = \frac{V_n^+ + V_n^-}{Z_B^+} \]

(Assume passband) \( \alpha = 0 \) \( z = nd \)
**Terminated Periodic Structures**

At the load where \( n = N \)

\[
V_N = V_N^+ + V_N^-
\]

\[
I_N = \frac{V_N^+ + V_N^-}{Z_B^+ + Z_B^-}
\]

& also \( V_N = Z_L I_N \)

Hence

\[
V_N^+ + V_N^- = Z_L \left( \frac{V_N^+}{Z_B^+} + \frac{V_N^-}{Z_B^-} \right) \rightarrow V_N^+ \left( 1 - \frac{Z_L}{Z_B^+} \right) = V_N^- \left( \frac{Z_L}{Z_B^-} - 1 \right)
\]

\[
\frac{V_N^-}{V_N^+} = \Gamma = \frac{Z_L/Z_B^- - 1}{Z_L/Z_B^+ - 1}
\]

\( (A=D) \) in case of symmetrical cells

\[
Z_B^+ = -Z_B^- = Z_B
\]

To avoid terminal reflections you must have

\[
Z_L = Z_B
\]

---

**Example 9.1**

A periodic loaded line, if \( Z_0 = 50 \, \Omega \), \( d = 1.0 \, \text{cm} \), and \( C_0 = 2.666 \, \text{pF} \), sketch the \( k-\beta \) diagram and compute the propagation constant, phase velocity, and Bloch impedance at \( f = 3.0 \, \text{GHz} \). Assume \( k = k_0 \).

\[
\cdot \cdot \cdot C_0 \quad \cdot \cdot \cdot C_0 \quad Z_0 \cdot k \quad \cdot \cdot \cdot C_0 \quad \cdot \cdot \cdot C_0 \quad C_0 \quad \cdot \cdot \cdot C_0 \quad \cdot \cdot \cdot
datale

**Answer**

\[
\cos \beta d = \cos \theta - \frac{b}{2} \sin \theta \quad \theta = k_d = k_s d
\]

\[
\cos \beta d = \cos k_0 d - \frac{b}{2} \sin k_0 d \quad \text{b is normalized to } Y_0 \quad b = \frac{\omega C_0}{Y_0} \quad \text{&} \quad Z_0 = \frac{1}{Y_0}
\]

\[
\cos \beta d = \cos k_0 d - \frac{\omega C_0 Z_0^2}{2} \sin k_0 d \quad \text{Note } k_0 d \text{ is function of } \omega
\]

So we will rewrite the function in term of \( k_0 d \)
Example 9.1

\[ \omega = k_o c = k_o d \frac{c}{d} \quad \cos \beta d = \cos k_o d - k_o d \left( \frac{C_o Z_o c}{2d} \right) \sin k_o d \]

\[ \frac{C_o Z_o c}{2d} = \frac{2.66 \times 10^{-12} \times 50 \times 3 \times 10^8}{2 \times 1 \times 10^{-2}} \cong 2 \]

\[ \cos \beta d = \cos k_o d - 2k_o d \sin k_o d \]

Passband \[ |\cos k_o d - 2k_o d \sin k_o d| < 1 \]

Stopband \[ |\cos k_o d - 2k_o d \sin k_o d| > 1 \]

The above equation can be evaluated numerically for given values of \( k_o d \)

\[ k_o d = \frac{\omega}{c} \quad \text{Since } c \text{ and } d \text{ are constants you are plotting against frequency} \]

Example 9.1
Example 9.1

At 3.0 GHz, we have

\[ k_o d = \frac{2\pi (3\times10^9)}{3\times10^8} (0.01) = 0.6283 = 36^\circ \]

\[ \cos \beta d = \cos (36) - 2(0.6283) \sin (36) \rightarrow \beta d = 1.5 \rightarrow \beta = 150 \]

\[ v_p = \frac{k_o c}{\beta} = \frac{0.6283}{1.5} = 0.42 c \]

**Bloch impedance**

\[ Z_B = \frac{BZ_o}{\sqrt{A^2 - 1}} \]

\[ b = \frac{\omega C_o Z_o}{2} = 1.256 \]

\[ \theta = k_o d = 0.6283 \]

\[ A = \cos \theta - \frac{b}{2} \sin \theta = 0.0707 \]

\[ B = j \left( \sin \theta + \frac{b}{2} \cos \theta - \frac{b}{2} \right) = j0.3479 \]

\[ Z_B = \frac{BZ_o}{\sqrt{A^2 - 1}} = \frac{(j0.3479)(50)}{j\sqrt{1-(0.0707)^2}} = 17.4 \Omega \]

---

**Image impedances & Transfer functions of two port network**

*Find Image Impedance and transfer function if a two port network?*

**Image Impedance as function of the ABCD Parameters**

\[ Z_{in1} = \text{input impedance at port 1 when port 2 is terminated with } Z_{i2} \]

\[ Z_{i2} = \text{input impedance at port 2 when port 1 is terminated with } Z_{i1} \]

**Note** that the reference direction for the current at port 2 has been chosen according to the convention for transmission parameters.

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Image impedances & Transfer functions of two port network

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \begin{bmatrix} A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\]

\[
\text{hence } \frac{V_1}{I_1} = \frac{AV_2 + BI_2}{CV_2 + DI_2} = \frac{A}{C} \frac{V_2}{I_2} + \frac{B}{D}.
\]

Input impedance at port 1, with port 2 terminated with \(Z_{i2}\)

\[
Z_{in1} = \frac{V_1}{I_1} = \frac{AV_2 + BI_2}{CV_2 + DI_2} = \frac{A}{C} \frac{V_2}{I_2} + \frac{B}{D}.
\]

because \(V_2 = Z_{i2}I_2\)

\[
Z_{in1} = \frac{AZ_{i2} + B}{CZ_{i2} + D}.
\]

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Image impedances & Transfer functions of two port network

Solving for \(V_2\) and \(I_2\) by Inverting ABCD matrix

\[
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix} = \begin{bmatrix} A & B \\
C & D
\end{bmatrix}^{-1} \begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} = \frac{1}{AD - BC} \begin{bmatrix} D & -B \\
-C & A
\end{bmatrix} \begin{bmatrix}
V_1 \\
I_1
\end{bmatrix}.
\]

Reciprocal network

\(AD - BC = 1\)

\[
V_2 = DV_1 - BI_1, \quad I_2 = -CV_1 + AI_1.
\]

\[
Z_{in2} = \frac{V_2}{I_2} = \frac{-DV_1 - BI_1}{-CV_1 + AI_1} = \frac{D}{C} \frac{-V_1}{I_1} + \frac{B}{A} = \frac{DZ_{i1} + B}{CZ_{i1} + A}.
\]

Due to the current direction

\[
Z_{i1} = \frac{-V_1}{I_1}.
\]

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Image impedances & Transfer functions of two port network

We desire that

\[ Z_{m1} = Z_{i1} \quad \text{&} \quad Z_{m2} = Z_{i2} \]

From

\[ Z_{m1} = Z_{i1} \]
\[ Z_{m2} = Z_{i2} \]

\[ Z_{i1} (CZ_{i2} + D) = AZ_{i2} + B \]
\[ Z_{i2} (CZ_{i1} + A) = DZ_{i1} + B \]

\[ Z_{i1} D - Z_{i2} A = Z_{i2} A - Z_{i1} D \]

Subtracting both equations

\[ 2Z_{i1} D = 2Z_{i2} A \]
\[ Z_{i1} = \frac{A}{D} Z_{i2} \]

\[ \frac{C D}{A} Z_{i1} + \frac{D}{Z_{i1}} = \frac{C D}{A} Z_{i2} + \frac{D}{Z_{i2}} \]
\[ \frac{C D}{A} Z_{i1} = \frac{C D}{A} Z_{i2} \]
\[ Z_{i1} = \frac{B}{C} Z_{i2} \]

& since

\[ Z_{i2} = \frac{D}{A} Z_{i1} \]

If symmetrical network

\[ A = D \]
\[ Z_{i1} = \frac{B}{C} Z_{i2} \]
\[ Z_{i1} = Z_{i2} \]

Image Impedance as function of the ABCD Parameters

Image impedances & Transfer functions of two port network

The transfer function of the network in term of ABCD

\[ V_1 = AV_2 + BI_2 \]
\[ V_2 = A + B \frac{I_2}{Z_{i2}} = A + B \frac{V_2}{Z_{i2}} \]
\[ I_2 = \frac{V_2}{Z_{i2}} \]

\[ \frac{V_1}{V_2} = A + B \frac{1}{Z_{i2}} \]
\[ = A + \frac{B D}{\sqrt{AC}} = A + \frac{\sqrt{ACB}}{\sqrt{D}} = A + \frac{\sqrt{ACB}}{\sqrt{D}} \]
\[ = \frac{A D + \sqrt{BC}}{D} \]

Note:

\[ \frac{x^2 - y^2}{x + y} = (x - y)(x + y) = x - y \]

Note:

For a symmetrical T or π networks the coefficient is unity

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Image impedances & Transfer functions of two port network

\[ \frac{V_2}{V_1} = \frac{D}{A} \left( \sqrt{AD} - \sqrt{BC} \right) \]

(Ratio)
For reciprocal networks this factor can be equal to unity.

Propagation factor of the network

\[ e^{-\gamma} = \frac{V_2}{V_1} = \left( \sqrt{AD} - \sqrt{BC} \right) \]

Let \( e^{-\gamma} = \sqrt{AD} - \sqrt{BC} \)
\[ \gamma = \alpha + j\beta \]

\[ e^{\gamma} = \left( \sqrt{AD} - \sqrt{BC} \right) \]
\[ AD - BC = 1 \]
\[ e^{\gamma} = \frac{1}{\left( \sqrt{AD} - \sqrt{BC} \right)} = \sqrt{AD} + \sqrt{BC} \]
\[ x^2 - y^2 = (x-y)(x+y) \]
\[ \frac{x+y}{x+y} = x - y \]

Note:

\[ \cosh \gamma = \frac{e^{\gamma} + e^{-\gamma}}{2} = \frac{\sqrt{AD} + \sqrt{BC} + \sqrt{AD} - \sqrt{BC}}{2} = \sqrt{AD} \]

\[ \cosh \gamma = \sqrt{AD} \]

Symmetrical T or \( \pi \) networks can be used to design the filters

\( \pi \) network

\( T \) network

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Next step is to develop the low-pass and high-pass filter section.

- **Low Pass Response**
- **Highpass Response**