Special cases of linear programming

- Infeasible solution
- Multiple solution (infinitely many solution)
- Unbounded solution
- Degenerated solution
Notes on the Simplex tableau

1. The intersection of any basic variable with itself is always one and the rest of the column is zeroes.

2. The objective function row (Z row) is always in terms of the nonbasic variables. This means that under any basic variable (in any tableau) there is a zero in the Z row. For the non basic there is no condition (it can take any value in this row).

3. If there is a zero under one or more nonbasic variables in the last tableau (optimal solution tableau), then there is a multiple optimal solution.

4. When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeroes), then the solution is unbounded.
5. If there is a tie (more than one variables have the same most negative or positive) in determining the entering variable, choose any variable to be the entering one.

6. If there is a tie in determining the leaving variable, choose any one to be the leaving variable. In this case a zero will appear in RHS column; therefore, a “cycle” will occur, this means that the value of the objective function will be the same for several iterations.

7. A Solution that has a basic variable with zero value is called a “degenerate solution”.

8. If there is no Artificial variables in the problem, there is no room for “infeasible solution”
Solve the following linear programming problem by using the simplex method:

Min Z = 2X_1 + 3X_2

S.t.
\[ \frac{1}{2} X_1 + \frac{1}{4} X_2 \leq 4 \]
\[ X_1 + 3X_2 \geq 20 \]
\[ X_1 + X_2 = 10 \]
\[ X_1, X_2 \geq 0 \]
Big M method

Solution
Step 1: standard form
Min Z,
s.t.
Z – 2 X₁ – 3 X₂ – M A₁ -M A₂ = 0
½ X₁ + ¼ X₂ + S₁ = 4
X₁ + 3X₂ - S₂ + A₁ = 20
X₁ + X₂ + A₂ = 10
X₁, X₂, S₁, S₂, A₁, A₂ ≥ 0
Where: M is a very large number
M, a very large number, is used to ensure that the values of $A_1$ and $A_2$, …, and $A_n$ will be zero in the final (optimal) tableau as follows:

1. If the objective function is **Minimization**, then $A_1$, $A_2$, …, and $A_n$ must be added to the RHS of the objective function multiplied by a very large number ($M$).

**Example:** if the objective function is Min $Z = X_1 + 2X_2$, then the obj. function should be Min $Z = X_1 + X_2 + MA_1 + MA_2 + \ldots + MA_n$

OR

$$Z = X_1 - X_2 - MA_1 - MA_2 - \ldots - MA_n = 0$$
Big M method

Maximization

$A_1, A_2, \ldots, \text{ and } A_n$ must be subtracted from the RHS of the objective function multiplied by a very large number (M).

Example: if the objective function is $\text{Max } Z = X_1 + 2X_2$, then the obj. function should be $\text{Max } Z = X_1 + X_2 - MA_1 - MA_2 - \ldots - MA_n$

OR

\[ Z - X_1 - X_2 + MA_1 + MA_2 + \ldots + MA_n = 0 \]

N.B.: When the Z is transformed to a zero equation, the signs are changed
### Big M method

#### Initial tableau

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( A_1 )</th>
<th>( A_2 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>( Z )</td>
<td>-2</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>-M</td>
<td>-M</td>
<td>0</td>
</tr>
</tbody>
</table>

- Note that one of the simplex rules is violated, which is the basic variables \( A_1 \), and \( A_2 \) have a non zero value in the \( z \) row; therefore,
- this violation must be corrected before proceeding in the simplex algorithm as follows.
Big M method
Correct violations

- To correct this, the simplex algorithm, the elementary row operations are used as follows:

\[ \text{New (Z row)} = \text{old (z row)} \pm M \ (A_1 \text{ row}) \pm M \ (A_2 \text{ row}) \]

In our case, it will be positive since M is negative in the Z row, as following:

<table>
<thead>
<tr>
<th>Old (Z row):</th>
<th>-2</th>
<th>-3</th>
<th>0</th>
<th>0</th>
<th>-M</th>
<th>-M</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>M (A1 row):</td>
<td>M</td>
<td>3M</td>
<td>0</td>
<td>-M</td>
<td>M</td>
<td>o</td>
<td>20M</td>
</tr>
<tr>
<td>M (A2 row):</td>
<td>M</td>
<td>M</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>10M</td>
</tr>
<tr>
<td>New (Z row):</td>
<td>2M-2</td>
<td>4M-3</td>
<td>0</td>
<td>-M</td>
<td>0</td>
<td>0</td>
<td>30M</td>
</tr>
</tbody>
</table>

It becomes zero
Big M method
Initial Tableau

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>X₁</th>
<th>X₂</th>
<th>S₁</th>
<th>S₂</th>
<th>A₁</th>
<th>A₂</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>S₁</td>
<td>1/2</td>
<td>1/4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>A₁</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>A₂</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Z</td>
<td>2M-2</td>
<td>4M-3</td>
<td>0</td>
<td>-M</td>
<td>0</td>
<td>0</td>
<td>30M</td>
</tr>
</tbody>
</table>

- Since there is a positive value in the last row, this solution is not optimal
- The entering variable is X₂ (it has the most positive value in the last row)
- The leaving variable is A₁ (it has the smallest ratio)
Big M method
First iteration

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>5/12</td>
<td>0</td>
<td>1</td>
<td>1/12</td>
<td>-1/12</td>
<td>0</td>
<td>7/3</td>
</tr>
<tr>
<td>$X_2$</td>
<td>1/3</td>
<td>1</td>
<td>0</td>
<td>-1/3</td>
<td>1/3</td>
<td>0</td>
<td>20/3</td>
</tr>
<tr>
<td>$A_2$</td>
<td>2/3</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>-1/3</td>
<td>1</td>
<td>10/3</td>
</tr>
<tr>
<td>$Z$</td>
<td>2/3M-1</td>
<td>0</td>
<td>0</td>
<td>1/3M-1</td>
<td>1-4/3M</td>
<td>0</td>
<td>20+10/3M</td>
</tr>
</tbody>
</table>

- Since there is a positive value in the last row, this solution is not optimal
- The entering variable is $X_1$ (it has the most positive value in the last row)
- The leaving variable is $A_2$ (it has the smallest ratio)
This solution is optimal, since there is no positive value in the last row. The optimal solution is:

\( X_1 = 5, \ X_2 = 5, \ S_1 = \frac{1}{4} \)

\( A_1 = A_2 = 0 \) and \( Z = 25 \)
Note for the Big M method

- In the final tableau, if one or more artificial variables \((A_1, A_2, \ldots)\) still basic and has a nonzero value, then the problem has an infeasible solution.
- All other notes are still valid in the Big M method.
Penalty method

- Split the problem in two phases
- In the first phase minimize artificial equality variables to find an initial feasible point
- In the phase start with the feasible solution and run a simplex method
Two phase method

Min $Z,$
\[ s.t. \]
\[
Z - 2X_1 - 3X_2 - A_1 - A_2 = 0 \\
\frac{1}{2}X_1 + \frac{1}{4}X_2 + S_1 = 4 \\
X_1 + 3X_2 - S_2 + A_1 = 20 \\
X_1 + X_2 + A_2 = 10 \\
X_1, X_2, S_1, S_2, A_1, A_2 \geq 0
\]

Where: $M$ is a very large number

Min $G,$
\[ s.t. \]
\[
G - 0X_1 - 0X_2 - A_1 - A_2 = 0 \\
\frac{1}{2}X_1 + \frac{1}{4}X_2 + S_1 = 4 \\
X_1 + 3X_2 - S_2 + A_1 = 20 \\
X_1 + X_2 + A_2 = 10 \\
X_1, X_2, S_1, S_2, A_1, A_2 \geq 0
\]

To find a feasible starting solution
Two phase
Initial tableau

<table>
<thead>
<tr>
<th>Basic variable s</th>
<th>X_1</th>
<th>X_2</th>
<th>S_1</th>
<th>S_2</th>
<th>A_1</th>
<th>A_2</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_1</td>
<td>½</td>
<td>¼</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>A_1</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>A_2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

- Note that one of the simplex rules is violated, which is the basic variables A_1, and A_2 have a non zero value in the z row; therefore,
- this violation must be corrected before proceeding in the simplex algorithm as follows.
### Two phase

#### Initial tableau

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>½</td>
<td>¼</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>S₁</td>
<td>½</td>
<td>¼</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-----</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>A₁</td>
<td>-1/2</td>
<td>-3/2</td>
<td>0</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>-10</td>
</tr>
<tr>
<td></td>
<td>S₁</td>
<td>0</td>
<td>-5/4</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
</tr>
<tr>
<td>A₁</td>
<td>-1</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-20</td>
</tr>
<tr>
<td>A₂</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>10</td>
</tr>
<tr>
<td>A₂</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-10</td>
</tr>
<tr>
<td>A₁</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-20</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>Z</td>
</tr>
</tbody>
</table>
## Two phase
### First iteration

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0</td>
<td>-5/4</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>$S_1$</td>
<td>0</td>
<td>-5/4</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>-10</td>
</tr>
<tr>
<td>$Z$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>
## Two phase
### second iteration tableau

<table>
<thead>
<tr>
<th>Basic variable</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Z$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Degeneracy

\[
\text{maximize } z = 3x_1 + 9x_2 \\
x_1 + 4x_2 \leq 8 \\
x_1 + 2x_2 \leq 4 \\
x_1, x_2 \geq 0
\]

- A tie between minimum ratios can occur
- One of them can be chosen arbitrary
- When this happens at least one basic variable will be zero in the next iteration
- Indicates that the minimum occurred at intersection of more than two constraints (in two dimensional optimization)
Degeneracy (graphically)

\[ z = 3x_1 + 9x_2 \]

Optimal degenerate solution

\[ x_1 + 4x_2 \leq 8 \text{ (redundant)} \]

\[ x_1 + 2x_2 \leq 4 \]
Degeneracy comments

- It has two problems
  - Cycling or Circling
    - can cause a repetitive sequence of iterations
    - Solution slow down the optimization
  - It may cause a stop in the iterations (because the objective function does not enhance)
Sensitivity Analysis
Change of the input

- The input data of the model can change without changing the optimum point (at the intersection of given number of constraints)
  - Change can be in the input constraints
  - Change in the coefficients of variables in the objective function
- Change in this case will change the value of the optimum but will not change the optimum point
The Role of Sensitivity Analysis of the Optimal Solution

- Is the optimal solution sensitive to changes in input parameters?

The effective of this change is known as “sensitivity”

It can be defined as the change of the optimum value with the change of one of the input constraints or with the change of one of the Constraints coefficients

Can be used to further tune the problem
Sensitivity Analysis of Objective Function Coefficients.

- These optimal point will remain the same (at the intersection of a given number of constraints up to given range of change in the constraints or the coefficients or the objective function)

- Range of Optimality
  - The optimal solution will remain unchanged as long as
    - An objective function coefficient lies within its *range of optimality*
    - There are no changes in any other input parameters.
JOBCO produces products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2 a Unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenue per unit of products 1 and 2 respectively is 30$ and 20$. Each machine can operate only 8 hours a day.

\[
\text{maximize } z = 30x_1 + 20x_2 \\
\text{Subject to } \\
2x_1 + x_2 \leq 8 \\
x_1 + 3x_2 \leq 8 \\
x_1, x_2 \geq 0
\]
The hours of operation of machine 1 has increased by 1 hour.
The revenue of has changed.
The rate of revenue change with the change of increasing the operation hours of machine 1:

\[
\begin{align*}
  x_1 + 3x_2 &\leq 8 \\
  x_1 + 3x_2 &\leq 8 \\
  2x_1 + x_2 &\leq 9 \\
  2x_1 + x_2 &\leq 8
\end{align*}
\]

Sensitivity with respect to the first constraint:

\[
x_1 = 3.8, \quad x_2 = 1.4, \quad z = 143
\]

\[
x_1 = 3.2, \quad x_2 = 1.6, \quad z = 128
\]

Solve simplex twice:

Increase of profit per hour work: Sensitivity = $14$
Sensitivity with respect to change in the Constraints

\[
\begin{align*}
\text{maximize } z &= 30x_1 + 20x_2 \\
\text{Subject to} & \\
2x_1 + x_2 &\leq 8 \\
x_1 + 3x_2 &\leq 8 \\
x_1, x_2 &\geq 0
\end{align*}
\]

|    | \(x_1\) | \(x_2\) | \(s_1\) | \(s_2\) |  \\
|----|--------|--------|--------|--------|---
| \(z\) | 30     | 20     | 14     | 2      | 128 |

Shadow prices
Example question that can be answered using sensitivity

- If the company can increase the hours of only one machine which one to choose?
- If the cost per hour increase in any machine would be 10$/hour would you still increase the machine working hour?
- If machine one working hours is increase by three hours how much would the profit increase.
- If machine 1 working hours are increased more than 20 minutes what would happen?
Determine range of optimality

$maximize \, z = 30x_1 + 20x_2$

Subject to

\[2x_1 + x_2 + s_1 + D_1 = 8\]
\[x_1 + 3x_2 + s_2 + D_2 = 8\]
\[x_1, x_2 \geq 0\]
The effects of changes in an objective function coefficients on the optimal solution.
Range of optimality

- find the range of optimality for an objectives function coefficient by determining the range of values that gives a slope of the objective function line between the slopes of the binding constraints.
Range of optimality for C1 is found by solving the following for C1:

\[-\frac{2}{1} \leq -\frac{C1}{5} \leq -\frac{3}{4}\]

\[3.75 \leq C1 \leq 10\]
Range optimality for Zapper, and coefficient per dozen space rays is $C_1 = 8$

Thus the slope of the objective function line can be expressed as $-\frac{8}{C_2}$
Range of optimality for C2 is found by solving the following for C2:

\[-\frac{2}{1} \leq -\frac{8}{C2} \leq -\frac{3}{4}\]

\[4 \leq C2 \leq 10.667\]
Maximize $z = 3x_1 + 2x_2 + 5x_3$

\[ x_1 + 2x_2 + x_3 \leq 430 \text{ (Operation 1)} \]
\[ 3x_1 + 2x_3 \leq 460 \text{ (Operation 2)} \]
\[ x_1 + 4x_2 \leq 420 \text{ (Operation 3)} \]

$x_1, x_2, x_3 \geq 0$
<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>Solution</th>
</tr>
</thead>
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<tr>
<td>$z$</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
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</tr>
<tr>
<td>$x_2$</td>
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<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{4}$</td>
<td>0</td>
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<td>1</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
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</tr>
<tr>
<td>$x_6$</td>
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<td>0</td>
<td>0</td>
<td>$-2$</td>
<td>1</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>
Maximize \( z = 3x_1 + 2x_2 + 5x_3 \)

\[
x_1 + 2x_2 + x_3 \leq 430 + D_1 \quad \text{(Operation 1)}
\]
\[
3x_1 + 2x_3 \leq 460 + D_2 \quad \text{(Operation 2)}
\]
\[
x_1 + 4x_2 \leq 420 + D_3 \quad \text{(Operation 3)}
\]

\( x_1, x_2, x_3 \geq 0 \)
### Linear and non-linear optimization

Mohamed Ashour, German University in Cairo

#### Lecture NETW904

<table>
<thead>
<tr>
<th>Basic</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
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<tbody>
<tr>
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<td>230</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>$-2$</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>$-2$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Solution**
\[
x_2 = 100 + \frac{1}{2}D_1 - \frac{1}{4}D_2 \geq 0
\]
\[
x_3 = 230 + \frac{1}{2}D_2 \geq 0
\]
\[
x_6 = 20 - 2D_1 + D_2 + D_3 \geq 0
\]

\[
x_2 = 100 + \frac{1}{2}(50) - \frac{1}{4}(-20) = 130 > 0
\]
\[
x_3 = 230 + \frac{1}{2}(-20) = 220 > 0
\]
\[
x_6 = 20 - 2(50) + (-20) + (-10) = -110 < 0
\]
\[ x_2 = 100 + \frac{1}{2}D_1 \geq 0 \Rightarrow D_1 \geq -200 \]
\[ x_3 = 230 > 0 \]
\[ x_6 = 20 - 2D_1 \geq 0 \Rightarrow D_1 \leq 10 \]
\[
\begin{align*}
\{ & x_2 = 100 - \frac{1}{4}D_2 \geq 0 \Rightarrow D_2 \leq 400 \\
& x_3 = 230 + \frac{1}{2}D_2 \geq 0 \Rightarrow D_2 \geq -460 \\
& x_6 = 20 + D_2 \geq 0 \quad \Rightarrow D_2 \geq -20
\end{align*}
\Rightarrow -20 \leq D_2 \leq 400
\]
\[ x_2 = 100 > 0 \]
\[ x_3 = 230 > 0 \]
\[ x_6 = 20 + D_3 \geq 0 \]
\[
\begin{align*}
\{ & x_2 = 100 > 0 \\
& x_3 = 230 > 0 \quad \Rightarrow -20 \leq D_3 < \infty \\
& x_6 = 20 + D_3 \geq 0
\end{align*}
\]
### Linear and Non-Linear Optimization

By Mohamed Ashour, German University in Cairo

#### Lecture NETW 904

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>( 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 )</td>
<td>0</td>
<td>0</td>
<td>( 1 + \frac{1}{2}d_2 )</td>
<td>( 2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 )</td>
<td>0</td>
<td>1350 + 100d_2 + 230d_3</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( -\frac{1}{4} )</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( -\frac{1}{4} )</td>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( \frac{3}{2} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>230</td>
</tr>
<tr>
<td>( x_6 )</td>
<td>( -\frac{1}{4} )</td>
<td>0</td>
<td>0</td>
<td>( -2 )</td>
<td>1</td>
<td>1</td>
<td>20</td>
</tr>
</tbody>
</table>

---

\[ \begin{array}{ccccccc}
0 & 0 & -2 & 1 & 1 & 20 \\
\end{array} \]

---

\[ \begin{array}{ccccccc}
0 & 0 & 0 & d_2 & d_3 & d_1 \end{array} \]

<table>
<thead>
<tr>
<th>Basic</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_6 )</th>
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<td>1</td>
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<td>0</td>
<td>1350</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>( x_2 )</td>
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<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( -\frac{1}{4} )</td>
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</tr>
<tr>
<td>( d_3 )</td>
<td>( x_3 )</td>
<td>( \frac{3}{2} )</td>
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<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>( x_6 )</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( -2 )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
### Reduced cost for $x_1 = 4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1$

<table>
<thead>
<tr>
<th>Left column</th>
<th>$x_1$</th>
<th>$(x_1\text{-column} \times \text{left-column})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>$4 \times 1$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$-\frac{1}{4}$</td>
<td>$-\frac{1}{4}d_2$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{3}{2}d_3$</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>$2 \times 0$</td>
</tr>
</tbody>
</table>

$4 - \frac{1}{4}d_2 + \frac{3}{2}d_3 - d_1 \geq 0$

$1 + \frac{1}{2}d_2 \geq 0$

$2 - \frac{1}{4}d_2 + \frac{1}{2}d_3 \geq 0$
Dual Problem

Primal problem

variables
\[ x_j \] variable \( j \)

objective
\[ \text{minimize } \sum_j c_j x_j \]

constraints
\[ \sum_j a_{ij} x_j \geq b_i, \]
\[ x_j \geq 0, \]

\[ j = 1, 2, \ldots, n \]

Dual Problem

variables
\[ \pi_i \] dual variable \( i \)

objective
\[ \text{maximize } \sum_i b_i \pi_i \]

constraints
\[ \sum_i a_{ij} \pi_i \leq c_j, \]
\[ \pi_i \geq 0, \]

\[ i = 1, 2, \ldots, m. \]
The primal problem has a bounded optimal solution \( x^* = (x_1^*, x_2^*, ..., x_n^*) \), if, and only if, the dual also has a bounded optimal solution \( \pi^* = (\pi_1^*, \pi_2^*, ..., \pi_m^*) \), and then the following holds:

\[
\sum_j c_j x_j^* = \sum_i b_i \pi_i^* .
\]
example

Minimize

\[18x_1 + 12x_2 + 2x_3 + 6x_4 \]
\[3x_1 + x_2 - 2x_3 + x_4 \geq 3\]
\[x_1 + 3x_2 - x_4 \geq 2\]

Maximize

\[3l_1 + 2l_2\]
\[3l_1 + l_2 \leq 18\]
\[l_1 + 3l_2 \leq 12\]
\[-2l_1 \leq 2\]
\[l_1 - l_2 \leq 12\]

4 Variables 2 constraints

2 Variables 4 constraints
Interpretation of the Dual problem (simple example)

A student wants to buy a snack such as to ensure he/she will get a minimums of certain foods

<table>
<thead>
<tr>
<th>Requirements</th>
<th>Chocolate</th>
<th>Sugar</th>
<th>Cream</th>
<th>Cheese</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brownie</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Cheesecake</td>
<td>0</td>
<td>4</td>
<td>5</td>
<td></td>
<td>80</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\min_{x_1, x_2} & \quad 50x_1 + 80x_2 \\
\text{subject to} & \quad 3x_1 \geq 6, \\
& \quad 2x_1 + 4x_2 \geq 10, \\
& \quad 2x_1 + 5x_2 \geq 8, \\
& \quad x_1, x_2 \geq 0,
\end{align*}
\]

Student Point of view
Minimize cost of snack

\[
\begin{align*}
\max_{u_1, u_2, u_3} & \quad 6u_1 + 10u_2 + 8u_3 \\
\text{subject to} & \quad 3u_1 + 2u_2 + 2u_3 \leq 50, \\
& \quad 4u_2 + 5u_3 \leq 80, \\
& \quad u_1, u_2, u_3 \geq 0.
\end{align*}
\]

Food supplier Point of view
Maximize profit given the minimum student requirement

Dual variables can be seen as the prices of the requirements
### Linear and non linear optimization

Mohamed Ashour, German University in Cairo

<table>
<thead>
<tr>
<th>Linear Equation</th>
<th>Primal in equation form</th>
<th>Dual variables</th>
</tr>
</thead>
</table>

Maximize $z = 5x_1 + 6x_2$

subject to

$\begin{align*}
  x_1 + 2x_2 &= 5 \\
  -x_1 + 5x_2 &\geq 3 \\
  4x_1 + 7x_2 &\leq 8 \\
  x_1 &\text{ unrestricted, } x_2 \geq 0
\end{align*}$

Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$

subject to

$\begin{align*}
  x_1^+ - x_1^- + 2x_2 &= 5 \\
  -x_1^- + x_1^+ + 5x_2 - x_3 &= 3 \\
  4x_1^- - 4x_1^+ + 7x_2 + x_4 &= 8 \\
  x_1^+, x_1^-, x_2, x_3, x_4 &\geq 0
\end{align*}$

**Dual Problem**

Minimize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$\begin{align*}
  y_1 - y_2 + 4y_3 &\geq 5 \\
  -y_1 + y_2 - 4y_3 &\geq -5 \\
  2y_1 + 5y_2 + 7y_3 &\geq 6 \\
  -y_2 &\geq 0 \\
  y_3 &\geq 0 \\
  y_1, y_2, y_3 &\text{ unrestricted}
\end{align*}$

$\Rightarrow (y_1 - y_2 + 4y_3 = 5)$

$\Rightarrow (y_1 \text{ unrestricted, } y_2 \leq 0, y_3 \geq 0)$
<table>
<thead>
<tr>
<th>Maximization problem</th>
<th>Minimization problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constraints</strong></td>
<td><strong>Variables</strong></td>
</tr>
<tr>
<td>$\leq$</td>
<td>$\leq 0$</td>
</tr>
<tr>
<td>$\geq$</td>
<td>$\geq 0$</td>
</tr>
<tr>
<td>$=$</td>
<td>Unrestricted</td>
</tr>
<tr>
<td><strong>Variables</strong></td>
<td><strong>Constraints</strong></td>
</tr>
<tr>
<td>$\geq 0$</td>
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