CIRCUIT ANALYSIS IN LAPLACE DOMAIN

“S” DOMAIN ANALYSIS
The Laplace Transform

The Laplace Transform of a function, $f(t)$, is defined as:

$$L[f(t)] = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{ts}ds$$
The Laplace Transform

An important point to remember:

\[ f(t) \iff F(s) \]

- The above is a statement that \( f(t) \) and \( F(s) \) are transform pairs.
- What this means is that for each \( f(t) \) there is a unique \( F(s) \) and for each \( F(s) \) there is a unique \( f(t) \).
The Laplace Transform

Laplace Transform of the unit step.

\[ L[u(t)] = \int_{0}^{\infty} e^{-st} \, dt = \frac{-1}{s} e^{-st} \bigg|_{0}^{\infty} \]

\[ L[u(t)] = \frac{1}{s} \]

The Laplace Transform of a unit step is: \[ \frac{1}{s} \]
The Laplace Transform

**Time Differentiation:** Making the previous substitutions gives,

\[
L \left[ \frac{df}{dt} \right] = f(t)e^{-st} \left|_{0}^{\infty} \int_{0}^{\infty} f(t) \left[ -se^{-st} \right] dt \right.
\]

\[
= 0 - f(0) + s \int_{0}^{\infty} f(t)e^{-st} dt
\]

So we have shown:

\[
L \left[ \frac{df(t)}{dt} \right] = sF(s) - f(0)
\]
The Laplace Transform

Final Value Theorem:

If the function \( f(t) \) and its first derivative are Laplace transformable and \( f(t) \) has the Laplace transform \( F(s) \),

and the \( \lim_{s \to \infty} sF(s) \) exists, then

\[
\lim_{s \to \infty} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)
\]

Again, the utility of this theorem lies in not having to take the inverse of \( F(s) \) in order to find out the final value of \( f(t) \) in the time domain. This is particularly useful in circuits and systems.
1.0 Energy Sources
2.0 Resistance

(v(t) = Ri(t))

Complex Frequency Domain

\[ V(s) = RI(s) \]
3.0 Inductor

\[ v_L(t) = L \frac{di(t)}{dt} \]

\[ L [V_L(t)] = V_L(S) \]

\[ V(s) = L [sI(s) - i(0^-)] = sLI(s) - Li(0^-) \]

\[ I(s) = \frac{1}{sL} V(s) + \frac{i(0^-)}{s} \]
4.0 Capacitor

\[ v_c(t) = \frac{1}{C} \int_0^t i(t) \, dt + v_c(0) \]

\[ i(t) = C \frac{dv(t)}{dt} \]

\[ I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-) \]

\[ V(s) = \frac{1}{sC} I(s) + \frac{v(0^-)}{s} \]
CIRCUIT ANALYSIS IN THE “S” DOMAIN
Summary

\[ Z(s) = \frac{V(s)}{I(s)} \]

\[ Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \]
Example

- Find $v_0(t)$ in the circuit, assuming zero initial conditions

\[ u(t) \quad \Rightarrow \quad \frac{1}{s} \]
\[ 1 \text{ H} \quad \Rightarrow \quad sL = s \]
\[ \frac{1}{3} \text{ F} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s} \]
Solution

\[
\frac{1}{s} = \left(1 + \frac{3}{s}\right) I_1 - \frac{3}{s} I_2
\]

\[
0 = -\frac{3}{s} I_1 + \left(s + 5 + \frac{3}{s}\right) I_2
\]

\[
I_1 = \frac{1}{3} (s^2 + 5s + 3) I_2
\]

\[
\frac{1}{s} = \left(1 + \frac{3}{s}\right) \frac{1}{3} (s^2 + 5s + 3) I_2 - \frac{3}{s} I_2
\]

\[
I_2 = \frac{3}{s^3 + 8s^2 + 18s}
\]

\[
V_o(s) = s I_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}} \frac{\sqrt{2}}{(s + 4)^2 + (\sqrt{2})^2}
\]

\[
v_o(t) = \frac{3}{\sqrt{2}} e^{-4t} \sin \sqrt{2}t \text{ V}, \quad t \geq 0
\]
Find $V_o(t)$ assuming $V_c(0) = 5 \text{ V}$
Write a node equation

\[ \frac{10}{s R_1} = \frac{V_o(s)}{R_2} - 5C + \frac{V_o(s)}{Cs} \]

\[ V_o(s) = \frac{\frac{10}{R_1} + 5s}{s + \frac{1}{R_2C}} = \frac{10}{R_1} + \frac{5 - 10}{R_1} \]

Inverse Laplace transform:

\[ v_o(t) = 10 \frac{R_2}{R_1} + \left( 5 - 10 \frac{R_2}{R_1} \right) e^{-t/R_2C} = 10 - 5e^{-1000t} \text{ V for } t > 0 \]
The Transfer function

- The *transfer function* is a key concept in signal processing because it indicates how a signal is processed as it passes through a network.
- It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis.
- The transfer function of a network describes how the output behaves in respect to the input.
- It specifies the transfer from the input to the output in the $s$ domain, assuming no initial energy.
The Transfer function

- The transfer function is also known as the network function.

The transfer function $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

- There are four possible transfer functions:

$$H(s) = \text{Voltage gain} = \frac{V_o(s)}{V_i(s)}$$

$$H(s) = \text{Current gain} = \frac{I_o(s)}{I_i(s)}$$

$$H(s) = \text{Impedance} = \frac{V(s)}{I(s)}$$

$$H(s) = \text{Admittance} = \frac{I(s)}{V(s)}$$
Application

- Determine the Transfer Function $H(s) = \frac{V_o(s)}{I_o(s)}$ of the circuit

By current division,

\[ I_2 = \frac{(s + 4)I_o}{s + 4 + 2 + 1/2s} \]

\[ V_o = 2I_2 = \frac{2(s + 4)I_o}{s + 6 + 1/2s} \]

\[ H(s) = \frac{V_o(s)}{I_o(s)} = \frac{4s(s + 4)}{2s^2 + 12s + 1} \]
Application

find: (a) the transfer function \( H(s) = \frac{V_o}{V_i} \),
(b) the response when \( v_i(t) = u(t) \) V

(a) Using voltage division,

\[
V_o = \frac{1}{s + 1} V_{ab}
\]

But

\[
V_{ab} = \frac{1 \parallel (s + 1)}{1 + 1 \parallel (s + 1)} V_i = \frac{(s + 1)/(s + 2)}{1 + (s + 1)/(s + 2)} V_i
\]

or

\[
V_{ab} = \frac{s + 1}{2s + 3} V_i
\]

Substituting Eq. (15.17.2) into Eq. (15.17.1) results in

\[
V_o = \frac{V_i}{2s + 3}
\]
(b) When \( v_i(t) = u(t), \ V_i(s) = 1/s, \) and
\[
V_o(s) = H(s)V_i(s) = \frac{1}{2s(s + \frac{3}{2})} = \frac{A}{s} + \frac{B}{s + \frac{3}{2}}
\]
where
\[
A = sV_o(s) \bigg|_{s=0} = \frac{1}{2(s + \frac{3}{2})} \bigg|_{s=0} = \frac{1}{3}
\]
\[
B = \left( s + \frac{3}{2} \right) V_o(s) \bigg|_{s=-3/2} = \frac{1}{2s} \bigg|_{s=-3/2} = -\frac{1}{3}
\]
Hence, for \( v_i(t) = u(t), \)
\[
V_o(s) = \frac{1}{3} \left( \frac{1}{s} - \frac{1}{s + \frac{3}{2}} \right)
\]
and its inverse Laplace transform is
\[
v_o(t) = \frac{1}{3} \left( 1 - e^{-3t/2} \right) u(t) \ \text{V}
\]
SINUSOIDAL FREQUENCY ANALYSIS

\[ B_0 \cos(\omega t + \theta) \rightarrow H(s) \rightarrow B_0 |H(j\omega)| \cos(\omega t + \theta + \angle H(j\omega)) \]

Circuit represented by network function
Application

a) Calculate the transfer function $V_o/V_i$

b) if $V_i = 2\cos(400t)\ V$, what is the steady state expression of $V_o$

**solution**

P 13.77  [a] $H(s) = \frac{-Z_f}{Z_i}$

$Z_f = \frac{(1/C_f)}{s + (1/R_fC_f)} = \frac{10^8}{s + 1000}$

$Z_i = \frac{R_i[s + (1/R_iC_i)]}{s} = \frac{10,000(s + 400)}{s}$

$H(s) = \frac{-10^4s}{(s + 400)(s + 1000)}$

[b] $H(j400) = \frac{-10^4(j400)}{(400 + j400)(1000 + j400)} = 6.565/-156.8^\circ$

$v_o(t) = 13.13\ \cos(400t - 156.8^\circ)\ V$
"That's all folks!"