ANTENNA ARRAYS
1-Why antenna arrays?

The array has 5 degrees of freedom which help in configuring some parameters

1. The element pattern
2. The amplitude excitation of the individual element
3. The phase excitation of the individual element
4. The relative displacement between elements
5. The geometrical distribution of the array, linear, rectangular, circular, spherical, …
N Elements linear arrays: Uniform amplitude and spacing

An array of identical elements fed with identical magnitude and a progressive phase. The array factor can be obtained by considering the elements to be point sources then multiplied times the element pattern

\[ AF = 1 + e^{j(kd \cos \theta + \beta)} + e^{j2(kd \cos \theta + \beta)} + \ldots + e^{j(N-1)(kd \cos \theta + \beta)} \]

\[ AF = \sum_{n=1}^{N} e^{j(n-1)(kd \cos \theta + \beta)} \]

which can be written as:

\[ AF = \sum_{n=1}^{N} e^{j(n-1)\psi} \]

where \( \psi = (kd \cos \theta + \beta) \)
\[ AF = 1 + e^{j\psi} + e^{j2\psi} + ..... + e^{j(N-1)\psi} \]  

(1)

*Multiplying both sides by \( e^{j\psi} \):*  

\[ AF e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + ..... + e^{j(N-1)\psi} + e^{jN\psi} \]  

(2)

*Subtracting (1) (2):

\[ AF(e^{j\psi} - 1) = (-1 + e^{jN\psi}) \]

\[ AF = \left[ \frac{e^{jN\psi} - 1}{e^{j\psi} - 1} \right] = e^{j\left(\frac{N-1}{2}\right)\psi} \left[ \frac{e^{j\left(\frac{N}{2}\right)\psi} - e^{-j\left(\frac{N}{2}\right)\psi}}{e^{j\left(\frac{1}{2}\right)\psi} - e^{-j\left(\frac{1}{2}\right)\psi}} \right] \]

\[ = e^{j\left(\frac{N-1}{2}\right)\psi} \left[ \frac{\sin\left(\frac{N}{2}\right)\psi}{\sin\left(\frac{1}{2}\right)\psi} \right] \]

*If the reference point is the physical center of the array, the array factor reduces to:*

\[ AF = \left[ \frac{\sin\left(\frac{N}{2}\right)\psi}{\sin\left(\frac{1}{2}\right)\psi} \right] \]
For small values of $\psi$, the above expression can be approximated by:

$$AF = \frac{\sin\left(\frac{N\psi}{2}\right)}{\frac{1}{2}\psi}$$

The maximum value of the array factor is equal to $(N)$, to normalize the array factor, so that the maximum value of each is equal to unity,

$$AF_n = \frac{\sin\left(\frac{N\psi}{2}\right)}{N \sin\left(\frac{1}{2}\psi\right)}$$

For small value of $\psi$ it is approximated by:

$$AF_n \approx \frac{\sin\left(\frac{N\psi}{2}\right)}{\frac{N}{2} \psi}$$

$$\psi = (kd \cos \theta + \beta)$$
**Maximum**
The maximum values occur when:

\[ \frac{\psi}{2} = \frac{1}{2} (kd \cos \theta + \beta)_{\theta = \theta_n} = \pm m \pi \]

\[ \theta_m = \cos^{-1} \left[ \frac{\lambda}{2\pi d} (- \beta \pm 2m \pi) \right] \quad m = 0, 1, 2, \ldots \]

\[ m = 0 \quad \text{for main lobe} \]

\[ m = 1, 2, \ldots \quad \text{for grating lobes} \]

**Nulls**
The null values of the array occur when:

\[ \sin \left( \frac{N}{2} \psi \right) = 0 \]

\[ \frac{N}{2} \psi_{\theta = \theta_n} = \pm n \pi \]

\[ \theta_n = \cos^{-1} \left[ \frac{\lambda}{2\pi d} (- \beta \pm \frac{2n}{N} \pi) \right] \quad n = 1, 2, 3, \ldots \]

\[ n \neq N, 2N, 3N, \ldots \]

**Maxima of minor lobes**
They occur approximately when:

\[ \sin \left( \frac{N}{2} \psi \right) = \pm 1 \]

or

\[ \frac{N}{2} (kd \cos \theta + \beta)_{\theta = \theta_s} = \pm \left( \frac{2s + 1}{2} \right) \pi \]

\[ \theta_s = \cos^{-1} \left[ \frac{\lambda}{2\pi d} (- \beta \pm \left( \frac{2s + 1}{2} \right) \pi) \right] \quad s = 1, 2, 3, \ldots \]

or

\[ \frac{N}{2} - \sin^{-1} \left[ \frac{\lambda}{2\pi d} (- \beta \pm \left( \frac{2s + 1}{2} \right) \pi) \right] \quad s = 1, 2, 3, \ldots \]

for \( d \gg \lambda \)

\[ \theta_s = \frac{\pi}{2} - \left[ \frac{\lambda}{2\pi d} (- \beta \pm \left( \frac{2s + 1}{2} \right) \pi) \right] \]
(a) Broadside array

The maximum radiation of any array is directed normal to the axis of the array (broadside $\theta_o=90^\circ$) which occurs when:

\[ \psi = kd \cos \theta_o + \beta = 0 \quad \text{put} \quad \theta_o = 90^\circ \]

$\psi = \beta = 0 \quad d \rightarrow \text{can be of any value}$

\[ AF_n = \left[ \sin \left( \frac{N}{2} \psi \right) \right] \]

\[ D_o = \frac{2L}{\lambda} \quad L = (n-1)d \]

The total phase difference $\psi$:

1- current excitation phase $\beta=0$
2- path difference phase $(kd\cos\theta)=0$, $\psi = 0$, i.e.,
constructive normal to the array axis $\theta=90^\circ$
The separation is an important constraint you have to mind to avoid the onset of the grating lobes in broadside arrays.

If the spacing between the elements is chosen between $\lambda < d < 2\lambda$, then the maximum toward $\theta_0 = 0^\circ$ shifts toward the angular region $0^\circ < \theta < 90^\circ$ while the maximum toward $\theta_0 = 180^\circ$ shifts toward $90^\circ < \theta < 180^\circ$.

If $d = 2\lambda$, there are maxima toward $0^\circ, 60^\circ, 90^\circ, 120^\circ$ and $180^\circ$.

\begin{itemize}
  \item GRATING LOBES $d > \lambda/2$
  \item NO GRATING LOBES $d \leq \lambda/2$
\end{itemize}
**Endfire array**

The maximum radiation of any array is directed along to the axis of the array \((\theta_0=0^\circ \text{ or } \theta_0=180^\circ)\) which occurs when:

\[
\psi = kd \cos \theta + \beta \bigg|_{\theta=0^\circ} = kd + \beta = 0 \quad \Rightarrow \quad \beta = -kd
\]

\[
\psi = kd \cos \theta + \beta \bigg|_{\theta=180^\circ} = -kd + \beta = 0 \quad \Rightarrow \quad \beta = kd
\]

\[
D_o = \frac{4L}{\lambda}, \quad L = (n-1)d
\]

\[
AF_n = \frac{\sin\left(\frac{N}{2} \psi\right)}{N \sin\left(\frac{1}{2} \psi\right)}
\]

\[
\psi = (kd \cos \theta \pm kd)
\]

The total phase difference \(\psi\):

1- current excitation phase \(\beta=-kd\)
2- path difference phase \((kd\cos \theta) = kd\), \(\psi = 0\), i.e.,

constructive along the array axis \(\theta = 0\)
N=10 \hspace{1cm} d=\lambda/4

\beta = -kd = -\pi/2 \hspace{1cm} \theta_o = 0^\circ

\beta = kd = \pi/2 \hspace{1cm} \theta_o = 180^\circ
(c) Phased (scanning) array

The maximum radiation of the array is required to be oriented at an angle $\theta_0 (0^\circ \leq \theta_0 \leq 180^\circ)$. To accomplish this, the phase excitation $\beta$ between the elements must be adjusted so that:

$$\psi = kd \cos \theta + \beta \bigg|_{\theta = \theta_0} = kd \cos \theta_0 + \beta \Rightarrow \beta = -kd \cos \theta_0$$

Thus by controlling the progressive phase difference between the elements, the maximum radiation can be squinted in any desired direction to form a scanning array

$$AF_n = \begin{bmatrix} \frac{\sin \left( \frac{N}{2} \psi \right)}{N} \\ \frac{1}{N} \sin \left( \frac{1}{2} \psi \right) \end{bmatrix}$$

Maxima

$$\frac{\psi}{2} = \frac{1}{2} (kd \cos \theta + \beta) \bigg|_{\theta = \theta_0} = \pm m \pi$$

$\ m = 0, 1, 2, \ldots$

Nulls

$$\sin \left( \frac{N}{2} \psi \right) = 0 \quad \quad \quad \frac{N}{2} \psi \bigg|_{\theta = \theta_0} = \pm n \pi$$

$\ n = 1, 2, \ldots$

$\ n \neq N, 2N, \ldots$
The total phase difference $\psi$:

1- current excitation phase $\beta = -kd \cos \theta_0$

2- path difference phase $(kd \cos \theta) = kd \cos \theta_o$

$\psi = 0$, i.e.,

constructive along the direction of scanning to maximum at $\theta = \theta_o$

It is clear that the general array can be scanned to any direction within the visible region (-kd to kd). The control parameter is the phase progressive of the excitation current; $\beta$

To avoid the onset of the grating lobe while scanning:

$-kd > \beta_o - 2\pi$ $\quad \Rightarrow \quad cos \theta_o > 1 - \lambda / d$ $\quad$ for an array along z-axis
(d) Different array orientations

• Linear array oriented along z-axis

\[ AF_n = \begin{bmatrix} \sin\left(\frac{N}{2} \psi\right) \\ N \sin\left(\frac{1}{2} \psi\right) \end{bmatrix} \]

\[ \psi = (kd \cos \theta + \beta) \]
• **Linear array oriented along x-axis**

\[
AF_n = \begin{bmatrix}
\sin\left(\frac{N}{2}\psi\right) \\
N \sin\left(\frac{1}{2}\psi\right)
\end{bmatrix}
\]

\[
\psi = (kd \cos \gamma + \beta)
\]

\[
\cos \gamma = \hat{a}_x \cdot \hat{a}_r = \hat{a}_x \cdot \left(\hat{a}_x \sin \theta \cos \phi + \hat{a}_y \sin \theta \sin \phi + \hat{a}_z \cos \theta\right)
\]

\[
\cos \gamma = \sin \theta \cos \phi
\]

\[
\psi = (kd \sin \theta \cos \phi + \beta)
\]
**Linear array oriented along y-axis**

\[
AF_n = \begin{bmatrix}
\sin\left(\frac{N}{2} \psi\right) \\
N \sin\left(\frac{1}{2} \psi\right)
\end{bmatrix}
\]

\[
\psi = (kd \cos \gamma + \beta)
\]

\[
\cos \gamma = \hat{a}_y \cdot \hat{r} = \hat{a}_y \cdot (\hat{a}_x \sin \theta \cos \phi + \hat{a}_y \sin \theta \sin \phi + \hat{a}_z \cos \theta)
\]

\[
\cos \gamma = \sin \theta \sin \phi
\]

\[
\psi = (kd \sin \theta \sin \phi + \beta)
\]
N Elements linear arrays: Nonuniform amplitude and uniform spacing

**Broadside arrays** $\beta=\theta$, Array along z-axis

**Even**

**Odd**
• **Array factor for even number**

\[
(AF)_{2M} = \left( a_1 e^{j \frac{kd}{2} \cos \theta} + a_2 e^{-j \frac{kd}{2} \cos \theta} + \ldots + a_M e^{-j \frac{(2M-1)kd}{2} \cos \theta} \right)
\]

\[
2a_1 \cos \left( \frac{kd}{2} \cos \theta \right) + 2a_2 \cos \left( \frac{3kd}{2} \cos \theta \right) + \ldots + 2a_M \cos \left( \frac{(2M-1)kd}{2} \cos \theta \right)
\]

\[
(AF)_{2M} = 2 \sum_{n=1}^{M} a_n \cos \left( \frac{(2n-1)kd}{2} \cos \theta \right)
\]

\[
(AF)_{2M} \mid_{n=1}^{M} a_n \cos \left[ \frac{(2n-1)kd}{2} \cos \theta \right]
\]

Let: \[
\frac{kd}{2} \cos \theta = \frac{\pi}{\lambda} d \cos \theta = u
\]

\[
(AF)_{2M} \text{ (even)} = \sum_{n=1}^{M} a_n \cos \theta \left[ (2n-1)u \right]
\]

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Array factor for odd number

\[
(AF)_{2M+1} = 2a_1 + a_2 e^{jkd \cos \theta} + \ldots + a_{M+1} e^{jMkd \cos \theta} \\
= 2a_1 + 2a_2 \cos(kd \cos \theta) + \ldots + 2a_{M+1} \cos(Mkd \cos \theta)
\]

\[
(AF)_{2M+1} = \sum_{n=1}^{M+1} a_n \cos[(n-1)kd \cos \theta]
\]

Let: 
\[
k d \frac{\cos \theta}{2} = \frac{\pi}{\lambda} d \cos \theta = u
\]

\[
(AF)_{2M+1} \text{(odd)} = \sum_{n=1}^{M+1} a_n \cos \theta \left[2(n-1)u\right]
\]
Binomial broadside array $\beta = 0$

**Binomial expansion**

$$(a + b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + n(n-1)\frac{a^{n-2} b^2}{2!} + \cdots$$

$$(1+x)^{m-1} = (1)^{m-1}(x)^0\frac{0!}{0!} + (m-1)(+1)^{m-2} x^1\frac{1!}{1!} + \cdots$$

$$(1+x)^{m-1} = 1 + \frac{m-1}{m-1} x + \frac{(m-1)(m-2)}{2!} x^2 + \cdots$$

$m=1: (1+x)^0 = 1$\hspace{1cm} 1

$m=2: (1+x)^1 = 1 + (1)x$\hspace{1cm} 1 1

$m=3: (1+x)^2 = 1 + 2x + (1)x^2$\hspace{1cm} 1 2 1

$m=4: (1+x)^3 = 1 + 3x + 3x^2 + (1)x^3$\hspace{1cm} 1 3 3 1

**Pascal’s triangle**

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**Example N=5**

2M+1=5 $\Rightarrow$ M=2

2$a_1 = 6$ $\Rightarrow$ $a_1 = 3$

$a_2 = 4$

$a_3 = 1$

**Example N=10**

$\frac{1}{a_5} \frac{9}{a_4} \frac{36}{a_3} \frac{84}{a_2} \frac{126}{a_1}$

$a_1 = 126$

$a_2 = 84$

$a_3 = 96$

$a_4 = 9$

$a_5 = 1$

Excitation Coefficients
It’s observed that there are no minor lobes for the arrays with spacing of $\lambda/4$ and $\lambda/2$ between the elements.

Binominal arrays have very low level minor lobes nearly zeros, but they exhibit large beamwidth.

A major disadvantage of binominal arrays is the wider variation between the amplitudes excitation of the different elements of an array.

**Example**

$N=10$

- $d = \lambda/4$
- $d = \lambda/2$
- $d = 3\lambda/4$
- $d = \lambda$

One more maximum appears (grating lobes) due to large spacing $d=3\lambda/4$, $d=\lambda$. 

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**Dolph Tschebyscheff broadside array**

-Since the array factor of an even or odd number of elements is a summation of cosine terms, whose form is a polynomial as the Dolph-Chebyscheff polynomials, then the unknown coefficients of the array factor can be determined by equating the series representing the cosine terms of the array factor to the appropriate Chebyscheff polynomial.

The order of the polynomial should be one less than the total number of elements of the array (array of 10 elements means Chebyshev polynomial of order 9).

\[ T_m(z) = \cos[m \cos^{-1}(z)] \quad \text{for } -1 \leq z \leq 1 \] represents the region of the side lobes

\[ T_m(z) = \cosh[m \cosh^{-1}(z)] \quad \text{for } z < -1, z > 1 \] represents the region of the main beam

\[ T_m(z) = 2zT_{m-1}(z) - T_{m-2}(z) \] is the recursive formula for Chebyscheff polynomial

\[ \cosh^{-1}(y) = \ln\left(y \pm \sqrt{y^2 - 1}\right) \]

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \cos m\mu )</th>
<th>( T_m(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \cos u = \cos u )</td>
<td>( Z )</td>
</tr>
<tr>
<td>2</td>
<td>( \cos 2u = 2\cos^2 u - 1 )</td>
<td>( 2Z^2 - 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( \cos 3u = 4\cos^3 u - 3\cos u )</td>
<td>( 4Z^3 - 3Z )</td>
</tr>
<tr>
<td>4</td>
<td>( \cos 4u = 8\cos^4 u - 8\cos^2 u + 1 )</td>
<td>( 8Z^4 - 8Z^2 + 1 )</td>
</tr>
<tr>
<td>5</td>
<td>( \cos 5u = 16\cos^5 u - 20\cos^3 u + 5\cos u )</td>
<td>( 16Z^5 - 20Z^3 + 5Z )</td>
</tr>
<tr>
<td>6</td>
<td>( \cos 6u = 32\cos^6 u - 48\cos^4 u + 18\cos^2 u - 1 )</td>
<td>( 32Z^6 - 48Z^4 + 18Z^2 - 1 )</td>
</tr>
<tr>
<td>7</td>
<td>( \cos 7u = 64\cos^7 u - 112\cos^5 u + 56\cos^3 u - 7\cos u )</td>
<td>( 64Z^7 - 112Z^5 + 56Z^3 - 7Z )</td>
</tr>
<tr>
<td>8</td>
<td>( \cos 8u = 128\cos^8 u - 256\cos^6 u + 160\cos^4 u - 32\cos^2 u + 1 )</td>
<td>( 128Z^8 - 256Z^6 + 160Z^4 - 32Z^2 + 1 )</td>
</tr>
<tr>
<td>9</td>
<td>( \cos 9u = 256\cos^9 u - 576\cos^7 u + 432\cos^5 u - 120\cos^3 u + 9\cos u )</td>
<td>( 256Z^9 - 576Z^7 + 432Z^5 - 120Z^3 + 9Z )</td>
</tr>
</tbody>
</table>
1- All polynomial pass through the point (1,1)
2- For \(-1 \leq z \leq 1\), \(T_m(z)\) have values within -1 to +1
3- All roots occur within \(-1 \leq z \leq 1\), all maxima and minima have values +1 and -1 respectively
**Design procedures**

1- Select the appropriate array factor (even or odd)

2- Expand the array factor \{cos\mu=.....\}

3- Determine the point \(z_o\), such that \(T_m(z_o)=R_o\)

   * The order (m) of the Chebyscheff polynomial is always one less than the total number of the element

4- Substitute cos \(u=z/z_o\)

5- Equate the array factor to \(T_m(z)\)

6- Determine \(a_n\)
**Example:**
Design a broadside Tschebyscheff array of 10 elements with spacing \((d)\) between the elements. The major to minor lobe level is 26 dB. Find the excitation coefficients and form the array factor.

**Solution:**

\[
1 - (AF)_{2M} = \sum_{n=1}^{5} a_n \cos[(2n-1)u], \quad u = \frac{\pi d}{\lambda} \cos \theta
\]

\[
2 - (AF)_{10} = a_1 \cos u + a_2 \cos 3u + a_3 \cos 5u + a_4 \cos 7u + a_5 \cos 9u
\]

Substitute for \(\cos u, \cos 3u, \cos 5u, \cos 7u, \cos 9u\) by series expansion

\[
3 - R_o (dB) = 26 = 20 \log R_o \implies R_o = 20
\]

\[
\therefore T_9(z_o) = 20 = \cosh[n \cosh^{-1}(z_o)]
\]

\[
\therefore z_o = \cosh\left[\frac{1}{9} \cosh^{-1}(20)\right] = 1.0851
\]

Or

\[
z_o = \frac{1}{2} \left[ \left(R_o + \sqrt{R_o^2 - 1} \right)^{\frac{1}{9}} + \left(R_o - \sqrt{R_o^2 - 1} \right)^{\frac{1}{9}} \right]
\]
4 – Substitute  \( \cos u = \frac{z}{z_o} = \frac{z}{1.0851} \) in the \( AF \) (step 2)

5 – Compare (step 4) with \( T_9(z) \)

\[
(AF)_{10} = \frac{z}{z_o} [a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5] + \left( \frac{z}{z_o} \right)^3 [4a_2 - 20a_3 + 55a_4 - 120a_5] + \\
\left( \frac{z}{z_o} \right)^5 [16a_3 - 112a_4 + 432a_5] + \left( \frac{z}{z_o} \right)^7 [64a_4 - 57a_5] + \left( \frac{z}{z_o} \right)^9 [256a_5]
\]

\[
= 9z - 120z^3 + 432z^5 - 576z^7 + 256z^9
\]

\( \therefore \) \( a_5 = 2.086 \) \( a_4 = 2.8308 \) \( a_3 = 4.1184 \) \( a_2 = 5.2073 \) \( a_1 = 5.8377 \)

in normalized form:

\( a_5 = 1 \) \( a_4 = 1.357 \) \( a_3 = 1.974 \) \( a_2 = 2.496 \) \( a_1 = 2.798 \)

or

\( a_5 = 0.357 \) \( a_4 = 0.485 \) \( a_3 = 0.706 \) \( a_2 = 0.89 \) \( a_1 = 1 \)

then \( (AF)_{10} = 2.798 \cos u + 2.496 \cos 3u + 1.974 \cos 5u + 1.357 \cos 7u + \cos 9u \)
**MXN Elements planner arrays: Nonuniform amplitude and uniform spacing**

The array factor for this planar array with its maximum along $\theta_0, \phi_0$, for an even number of elements in each direction can be written as:

$$[AF(\theta, \phi)]_{M \times N} = 4 \sum_{m=1}^{M/2} \sum_{n=1}^{N/2} w_{mn} \cos[(2m-1)u] \cos[(2n-1)v]$$

where

$$u = \frac{\pi d_x}{\lambda} \left(\sin \theta \cos \phi - \sin \theta_0 \cos \phi_0\right)$$

$$v = \frac{\pi d_y}{\lambda} \left(\sin \theta \sin \phi - \sin \theta_0 \sin \phi_0\right)$$

$w_{mn}$ is the amplitude excitation of the individual element.

For separable distributions $w_{mn} = w_m w_n$.

Number of excitation values need to be computed are $M+N$.

For nonseparable distributions $w_{mn} \neq w_m w_n$.

Number of excitation values need to be computed are $M \times N$.

i.e., $w_{mn}$ is a 1 x $N$ vector for a linear array and a $M \times N$ matrix for a planar array.

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• The amplitude coefficients $a_n$ or $w_n$ control primarily the shape of the pattern and the major-to-minor lobe level
• Tapered amplitude distributions exhibit wider beamwidth but lower sidelobe
• The phase excitations control primarily the scanning capabilities of the array

Therefore, an antenna designer can choose different amplitude distributions and different phase excitation to conform to the application specifications

In smart antenna systems the objectives of a DSP are to estimate:
1. The direction of arrival (DOA) of all impinging signals using DOA algorithms, and
2. The appropriate weights using the adaptive beam forming algorithms to ideally steer the maximum radiation of the antenna pattern toward the SOI and to place nulls toward the SNOI